

Regular representations of vertex operator algebras, I

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1 Introduction

In a previous study [Li3], the physical super selection principle in vertex operator algebra theory seems to tell us that for a vertex operator algebra V , one should be able to obtain each irreducible V -module from the adjoint module V by changing certain things. To a certain extent, what we are expecting is an analogue of the orbital construction of representations for a Lie group [Ki]. This was our original motivation for this study on regular representations of vertex operator algebras.

This is the first paper in a series for this study. In this paper, given a module W for a vertex operator algebra V and a nonzero complex number z we construct a canonical (weak) $V \otimes V$ -module $\mathcal{D}_{P(z)}(W)$ (a subspace of W^* depending on z). We prove that for V -modules W, W_1 and W_2 , a $P(z)$ -intertwining map of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$ ([H3], [HL0-3]) exactly amounts to a $V \otimes V$ -homomorphism from $W_1 \otimes W_2$ into $\mathcal{D}_{P(z)}(W)$. Using Huang and Lepowsky's one-to-one linear correspondence between the space of intertwining operators and the space of $P(z)$ -intertwining maps of the same type we obtain a canonical linear isomorphism from the space $\mathcal{V}_{W_1 W_2}^{W'}$ of intertwining operators of the indicated type to $\text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W))$. In the case that $W = V$, we obtain a decomposition of Peter-Weyl type for $\mathcal{D}_{P(z)}(V)$, which are what we call the regular representations of V .

Let us start with the classical Peter-Weyl theory. Let G be a group G and $F(G, \mathbb{C})$ be the space of complex-valued functions on G . Then $F(G, \mathbb{C})$ is a $G \times G$ -module with

$$((g_1, g_2)f)(g) = f(g_1^{-1}gg_2) \quad (1.1)$$

for $g_1, g_2, g \in G, f \in F(G, \mathbb{C})$. Furthermore, if G is a topological (Lie, algebraic) group, all continuous (\mathbb{C}^∞ , algebraic) functions form a $G \times G$ -submodule. Now let G be a compact Lie group and $C^0(G)$ be the space of continuous functions on G . A representative function on G is a continuous function which generates a finite-dimensional $1 \otimes G$ -submodule of $C^0(G)$, and the regular representation of G is the $G \times G$ -module $\mathcal{J}(G, \mathbb{C})$ of representative functions on G (cf. [BD]). Let U be an irreducible G -module. For $u \in U, u^* \in U^*$, we can view $u^* \otimes u$ as a function on G by

$$(u^* \otimes u)(g) = \langle u^*, gu \rangle \quad \text{for } g \in G.$$

Then $U^* \otimes U$ is embedded into $C^0(G)$ as a $G \otimes G$ -module. A theorem of Peter-Weyl type states that $\mathcal{J}(G, \mathbb{C})$ is a dense subspace of $C^0(G)$ and that $\mathcal{J}(G, \mathbb{C})$ is a direct sum of

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$U^* \otimes U$, where U runs through a complete set of representatives of equivalence classes of finite-dimensional irreducible G -modules.

To start vertex operator algebra theory, let us recall from [FLM] and [FHL] (see also [B]) the definition of vertex operator algebra, which will be the official definition for this paper. A *vertex operator algebra* is a \mathbb{Z} -graded vector space $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ such that $\dim V_{(n)} < \infty$ for all $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently small, equipped with a bilinear “vertex multiplication”

$$\begin{aligned} Y(\cdot, x) \cdot : \quad V \otimes V &\rightarrow V((x)) \\ u \otimes v &\mapsto Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v x^{-n-1} \end{aligned} \quad (1.2)$$

such that for $m, n, k \in \mathbb{Z}$, $u \in V_{(k)}$,

$$u_n V_{(m)} \subset V_{(m+k-n-1)}, \quad (1.3)$$

and such that the *Jacobi identity* holds for $u, v \in V$:

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2). \end{aligned} \quad (1.4)$$

It is equipped with a vector $\mathbf{1}$, called the *vacuum vector*, such that for $v \in V$,

$$Y(\mathbf{1}, x)v = v, \quad (1.5)$$

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v. \quad (1.6)$$

It is also equipped with a vector ω , called the *Virasoro element*, such that

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m+n, 0}(\text{rank} V) \quad (1.7)$$

for $m, n \in \mathbb{Z}$, where $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$ and $\text{rank} V$ is a complex number, called the *rank* of V , and such that for $v \in V_{(n)}$, $n \in \mathbb{Z}$,

$$Y(L(-1)v, x) = \frac{d}{dx} Y(v, x), \quad (1.8)$$

$$L(0)v = nv. \quad (1.9)$$

This completes the definition. It is well known ([FLM], [FHL]) that the Jacobi identity is equivalent to the following rationality, commutativity and associativity:

(rationality) For $u, v, w \in V$, $w' \in V'$, where $V' = \coprod_n V_{(n)}^*$, the formal series

$$\langle w', Y(u, x_1)Y(v, x_2)w \rangle \left(= \sum_{m, n \in \mathbb{Z}} \langle w', u_m v_n w \rangle x_1^{m-1} x_2^{-n-1} \right) \quad (1.10)$$

absolutely converges to a rational function of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{(x_1 - x_2)^k} \quad (1.11)$$

in the domain $|x_1| > |x_2| > 0$, where $g \in \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ and k is an integer only depending on u and v ; independent of w' and w ;

(commutativity) The formal series $\langle w', Y(v, x_2)Y(u, x_1)w \rangle$ absolutely converges to the same rational function $f(x_1, x_2)$ in the domain $|x_2| > |x_1| > 0$;

(associativity) The formal series

$$\langle w', Y(Y(u, x_0)v, x_2)w \rangle \left(= \sum_{m, n \in \mathbb{Z}} \langle w', (u_m v)_n w \rangle x_0^{-m-1} x_2^{-n-1} \right) \quad (1.12)$$

absolutely converges to the rational function $f(x_0 + x_2, x_2)$ in the domain $|x_2| > |x_0| > 0$. If we specialize x_2 to a nonzero complex number z , (1.10) gives rise to a meromorphic function on the sphere $CP^1 = \mathbb{C} \cup \{\infty\}$ with only three possible poles at $0, \infty$ and z while (1.12) gives rise to a meromorphic function on the sphere CP^1 with only three possible poles at $0, \infty$ and $-z$. The rationality, commutativity and associativity is the basis for Huang's geometric interpretation of a vertex operator algebra [H1].

For a vertex operator algebra V , we have the following fundamental results established in [FHL]. It was proved that $V \otimes V$ has a natural vertex operator algebra structure and that for any V -modules W_1 and W_2 , $W_1 \otimes W_2$ is a natural $V \otimes V$ -module; Let $W = \coprod_{h \in \mathbb{C}} W_{(h)}$ be a V -module and let $W' = \coprod_{h \in \mathbb{C}} W_{(h)}^*$, the restricted dual. For $v \in V$, define $Y'(v, x)$ to be an element of $(\text{End } W')[[x, x^{-1}]]$ by

$$\langle Y'(v, x)w', w \rangle = \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \rangle \quad (1.13)$$

for $w' \in W'$, $w \in W$. Then (W', Y') carries the structure of a V -module; For $v \in V$, set

$$Y^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \quad (1.14)$$

It was proved ([HL1], [FHL]) that Y^o satisfies the opposite Jacobi identity (2.12), so that the pair (W, Y^o) carries the structure of a right V -module. These facts should be compared with those mentioned in the second paragraph for a group.

Now we consider what the space of a regular representation should be. The classical analogue suggests that we consider "functions on V of the form $w' \otimes w$ " for $w' \in W'$, $w \in W$, where W is an irreducible V -module. By analogy we should view $w' \otimes w$ as a $\mathbb{C}[x, x^{-1}]$ -valued linear function on V by considering $\langle w', Y(v, x)w \rangle$ for $v \in V$. To get a \mathbb{C} -valued function on V , we use Huang and Lepowsky's idea to evaluate the formal variable x at a nonzero complex number z as in [H3] and [HL0-3], so that $w' \otimes w$ can be viewed as a linear functional on V by

$$(w' \otimes w)(v) = \langle w', Y(v, z)w \rangle \quad \text{for } v \in V. \quad (1.15)$$

Such functionals satisfy a certain nice rationality property as described below. For a linear functional α on V , we naturally extend α to a linear function on $V[[x, x^{-1}]]$. Then consider the formal series $\alpha(Y(u, x)v)$ with $u, v \in V$. If $\alpha = w' \otimes w$, we have

$$\alpha(Y(u, x)v) = \langle w', Y(Y(u, x)v, z)w \rangle,$$

which by associativity converges to a rational function in the domain $0 < |x| < |z|$ with only two possible (finite) poles at 0 and $-z$. Globally, from Y we obtain a linear map F from $W' \otimes W$ to V^* defined by

$$\langle F(w' \otimes w), v \rangle = \langle w', Y(v, z)w \rangle \quad (1.16)$$

for $w' \in W'$, $w \in V$, $v \in V$. This map F is nothing but a $Q(z)$ -intertwining map as known in [HL1]. Linear functionals α with this rationality property are called $Q(z)$ -linear functions and will be studied in [Li6].

In this paper we shall study the so-called $P(z)$ -linear functionals defined below. Notice that previously we basically used the canonical intertwining operator Y of type $\begin{pmatrix} W \\ VW \end{pmatrix}$. It was known ([FHL], [HL2]) that there are canonical linear isomorphisms from the space of intertwining operators of type $\begin{pmatrix} W \\ VW \end{pmatrix}$ to the space of intertwining operators of type $\begin{pmatrix} V' \\ W'W \end{pmatrix}$. Let \mathcal{Y} be an intertwining operator of type $\begin{pmatrix} V' \\ W'W \end{pmatrix}$. (One can make use of the vacuum vector to get a canonical one.) Then $\mathcal{Y}(w', z)w$ is a linear functional on V . (To be rigorous we have to choose a branch of log function to evaluate x at z for $\mathcal{Y}(w', x)w$, see Section 4, or [HL1]. Here we neglect this issue for this introduction.) This gives another way to view $w' \otimes w$ as a linear functional on V . This time, for $u, v \in V$, the formal series

$$\langle Y(u, x)\mathcal{Y}(w', z)w, v \rangle \quad (= \langle \mathcal{Y}(w', z)w, Y^o(u, x)v \rangle)$$

by commutativity converges to a rational function in the domain $|x| > |z|$ with only two possible poles at 0 and z , or a meromorphic function on CP^1 with only three possible poles at 0, z and ∞ . Globally, evaluated intertwining operators $\mathcal{Y}(\cdot, z)\cdot$ at $x = z$ are exactly captured by the notion of $P(z)$ -intertwining map ([H3], [HL0-3]). Then we consider linear functionals α on V such that for $u, v \in V$, the formal series $\alpha(u, v, x) := \alpha(Y^o(u, x)v)$ converges to a rational function of x with only two possible poles at 0 and z , and we call such linear functionals $P(z)$ -linear functionals on V and denote by $\mathcal{D}_{P(z)}(V)$ the space of all $P(z)$ -linear functionals. We define $Y^R(u, x)\alpha$ and $Y^L(u, x)\alpha$ by requiring $\langle Y^R(u, x)\alpha, v \rangle$ to be the formal Laurent series expansion of the rational function $\alpha(u, v, x)$ in the domain $0 < |x| < |z|$, and $\langle Y^L(u, x)\alpha, v \rangle$ to be the Laurent series expansion of the rational function $\alpha(u, v, x + z)$ in the domain $0 < |x| < |z|$. Our first main result is that Y^L and Y^R give rise to a structure of a weak $V \otimes V$ -module on $\mathcal{D}_{P(z)}(V)$ in the sense that all the axioms defining the notion of a module except those involving the grading hold.

Vertex operator algebras are similar to Lie algebras. One of the similarities is that ideals are always two-sided due to the skew-symmetry. Consequently, a simple vertex operator algebra itself is an irreducible module. Most of the time, the adjoint module and other modules can be equally treated. Here, we construct a weak $V \otimes V$ -module $\mathcal{D}_{P(z)}(W)$

for a general V -module W . Then, for any additional modules W_1 and W_2 , we identify the space $\mathcal{M}[P(z)]_{W_1 W_2}^{W'}$ of $P(z)$ -intertwining maps of the indicated type with $\text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W))$. If V is regular, i.e., any weak V -module is a direct sum of irreducible (ordinary) modules, then we obtain a decomposition of $\mathcal{D}_{P(z)}(W)$ into irreducible $V \otimes V$ -modules with the corresponding fusion rules as the multiplicities. In the case that $W = V$, we obtain a decomposition of $\mathcal{D}_{P(z)}(V)$ of Peter-Weyl type.

This paper is intimately related to [H3, HL0-3] and we use certain basic principles and techniques that have been explored herein. In the tensor product theory for a vertex operator algebra [H3, HL0-3], for *two* given V -modules W_1 and W_2 , a V -module “ $W_1 \otimes_{P(z)} W_2$ ” was constructed such that for any V -module W , a $P(z)$ -intertwining map of type $\binom{W'}{W_1 W_2}$ canonically gives rise to a V -homomorphism from “ $W_1 \otimes_{P(z)} W_2$ ” to W' . In the present work, by constructing a $V \otimes V$ -module $\mathcal{D}_{P(z)}(W)$ for *one* given V -module W , a $P(z)$ -intertwining map of type $\binom{W'}{W_1 W_2}$ becomes a homomorphism in the category of (weak) $V \otimes V$ -modules instead of the category of V -modules.

In a recent work [FM], Frenkel and Malikov obtained some interesting results by using Kazhdan-Lusztig tensoring and Harish-Chandra categories. The bimodule $\mathcal{D}_{P(z)}(W)$ is analogous to the Harish-Chandra bimodule associated to a module for a Lie algebra. We hope to study $\mathcal{D}_{P(z)}(V)$ in this aspect in a future publication. During this research we have noticed that regular representations for affine Lie algebras have been studied in [FP] and that certain results of Peter-Weyl type on Kac-Moody groups have been obtained in [KP]. Presumably, [FP], [KP] (in the affine case) and the present paper are closely related. We hope to study the connections in a future publication.

In the next paper [Li6] we shall define and study $\mathcal{D}_{Q(z)}(W)$ for a given V -module W and we shall study the connection between the regular representation $\mathcal{D}_{Q(z)}(V)$ and the modular invariance property of trace functions and correlation functions ([Z], [DLM3]).

This paper is organized as follows: In Section 2 we review the notion of the contragredient module and discuss certain related issues. In Section 3, we present $V \otimes V$ -module $\mathcal{D}_{P(z)}(W)$. In Section 4, we present the analogue of the Peter-Weyl theorem.

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2 Contragredient modules and some related issues

This section is preliminary. Here we review some basic notions and facts, including the notion of contragredient module and an extension of this. We shall discuss the notion of right module and prove certain results.

We shall use standard definitions and notions such as the notions of vertex operator algebra, (irreducible) module, homomorphism, for which we refer the reader to [FHL] and [FLM]. Since we here only deal with the representation theory of vertex operator algebras, we fix a vertex operator algebra V throughout this paper. We shall typically use $x, y, x_1, x_2, x_3, \dots$ for mutually commuting formal variables, use letters u, v for elements of V and use $w, w_{(1)}, w_{(2)}, \dots$ for elements of V -modules. We use \mathbb{Z} for the set of all integers, \mathbb{N} for the set of all nonnegative integers and \mathbb{C} for the field of complex numbers. We shall also use certain relatively new notions, which we recall next.

Definition 2.1 A *weak* V -module is a vector space W equipped with a linear map Y_W , called the *vertex operator map*, from V to $(\text{End}W)[[x, x^{-1}]]$ such that the *truncation condition*:

$$Y_W(v, x)w \in W((x)) \quad \text{for } v \in V, w \in W; \quad (2.1)$$

the *vacuum property*:

$$Y_W(\mathbf{1}, x) = 1 \quad (1 \text{ on the right being the identity operator on } W); \quad (2.2)$$

and the *Jacobi identity*:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2) \end{aligned} \quad (2.3)$$

for $u, v \in V$, hold on W .

Remark 2.2 It was proved in [DLM2] that the $L(-1)$ -derivative property:

$$Y_W(L(-1)v, x) = \frac{d}{dx} Y_W(v, x) \quad (2.4)$$

for $v \in V$ holds for a weak V -module W . Thus a weak V -module satisfies all the axioms for a V -module given in [FHL] and [FLM] except for those involving the $L(0)$ -grading.

Throughout this paper, a V -module always stands for an ordinary V -module unless it is labeled as a weak or generalized V -module.

Vertex operator algebra V is said to be *regular* [DLM2] if any weak V -module is a direct sum of irreducible (ordinary) V -modules. If V is regular, it was proved in [DLM2]

that V has only finitely many inequivalent irreducible modules. Examples of regular vertex operator algebras were given in [DLM2].

It is well known (cf. [B], [FFR], [Li1], [Li4], [MP]) that there is a Lie algebra $g(V)$ associated to V . More precisely,

$$g(V) = \hat{V}/d\hat{V}, \quad (2.5)$$

where

$$\hat{V} = V \otimes \mathbb{C}[t, t^{-1}] \quad \text{and} \quad d = L(-1) \otimes 1 + 1 \otimes \frac{d}{dt},$$

with the following Lie bracket:

$$[u(m), v(n)] = \sum_{i \geq 0} \binom{m}{i} (u_i v)(m + n - i) \quad (2.6)$$

for $u, v \in V$, $m, n \in \mathbb{Z}$, where $u(m) = u \otimes t^m + d\hat{V}$. Furthermore, $g(V)$ is a \mathbb{Z} -graded Lie algebra where $\deg u(m) = \text{wt} u - m - 1$ for homogeneous $u \in V$ and for $m \in \mathbb{Z}$.

Let W be a $g(V)$ -module. A vector w of W is said to be *restricted* if for every $v \in V$, $v(m)w = 0$ for m sufficiently large. Denote by W^{res} the space of all the restricted vectors of W . Since for any $u, v \in V$, $u_i v \neq 0$ for only finitely many nonnegative integer i , it follows from (2.6) that W^{res} is a submodule of W . A $g(V)$ -module W is said to be *restricted* if $W = W^{res}$. In general, it is clear that W^{res} is the unique maximal restricted submodule of W . On the other hand, because of the truncation condition any weak V -module W is a restricted $g(V)$ -module where $v(n)$ is represented by v_n for $v \in V$, $n \in \mathbb{Z}$.

We now recall the theory of *contragredient module* established in [FHL]. Let $W = \coprod_{h \in \mathbb{C}} W_{(h)}$ be a V -module and let $W' = \coprod_{h \in \mathbb{C}} W_{(h)}^*$, the *restricted dual* of W . For $v \in V$, $w' \in W'$, define

$$\langle Y'(v, x)w', w \rangle = \left\langle w', Y \left(e^{xL(1)} (-x^{-2})^{L(0)} v, x^{-1} \right) w \right\rangle \quad (2.7)$$

for $w \in W$. Then we have the following fundamental result due to Frenkel, Huang and Lepowsky ([FHL], Theorem 5.2.1 and Proposition 5.3.1).

Proposition 2.3 *The pair (W', Y') carries the structure of a V -module and $(W'', Y'') = (W, Y)$.*

Let W be a weak V -module for now. For $v \in V$, set (cf. [HL1])

$$Y^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \in (\text{End } W)[[x, x^{-1}]]. \quad (2.8)$$

(Note that $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$.) For instance,

$$Y^o(\mathbf{1}, x) = 1, \quad (2.9)$$

$$Y^o(\omega, x) = x^{-4}Y(\omega, x^{-1}) \quad (2.10)$$

because $L(0)\mathbf{1} = L(1)\mathbf{1} = 0$, $L(0)\omega = 2\omega$ and $L(1)\omega = 0$, where ω is the Virasoro element of V . Since $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$ for $v \in V$ and $Y(u, x^{-1})w \in W((x^{-1}))$ for any $u \in V$, we have

$$Y^o(v, x)w \in W((x^{-1})) \quad \text{for } w \in W. \quad (2.11)$$

It was observed in [HL1] that FHL's proof in [FHL] for Proposition 2.3 in fact proves the following *opposite Jacobi identity*:

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y^o(v, x_2)Y^o(u, x_1) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y^o(u, x_1)Y^o(v, x_2) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y^o(Y(u, x_0)v, x_2). \end{aligned} \quad (2.12)$$

As it was mentioned in [HL1] one should think the pair (W, Y^o) as a right weak V -module. Since we shall discuss a little more further, let us make a formal definition here: A *right weak V -module* is a vector space W equipped with a linear map Y_W from V to $(\text{End}W)[[x, x^{-1}]]$ such that for $u, v \in V$, $w \in W$,

$$Y_W(v, x)w \in W((x^{-1})), \quad (2.13)$$

$$Y_W(\mathbf{1}, x) = 1, \quad (2.14)$$

and such that the opposite Jacobi identity (2.12) with Y^o being replaced by Y_W holds.

Remark 2.4 Similar to Remark 2.2, for a right weak V -module (W, Y_W) we have

$$Y_W(L(-1)v, x) = \frac{d}{dx}Y_W(v, x) \quad (2.15)$$

for $v \in V$. From the opposite Jacobi identity we get

$$[Y_W(v, x), L(-1)] = Y_W(L(-1)v, x). \quad (2.16)$$

Combining (2.15) with (2.16) we get

$$[L(-1), Y_W(v, x)] = -\frac{d}{dx}Y_W(v, x). \quad (2.17)$$

Remark 2.5 Given a right weak V -module (W, Y_W) , using the opposite Jacobi identity and Remark 2.4 we get the Virasoro right module relations on W :

$$-[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\text{rank}V). \quad (2.18)$$

Remark 2.6 A *right V -module* is defined to be a right weak V -module W on which $L(0)$ semisimply acts such that the grading on W , given by the $L(0)$ -eigenspaces, satisfies the same two grading restrictions as those in defining the notion of a (left) V -module. Then $L(-n)$ are locally nilpotent on W for $n \geq 1$. (Recall that $L(n)$ are locally nilpotent on left modules.)

Let U be a vector space together with a linear map Y_U from the vertex operator algebra V to $(\text{End } U)[[x, x^{-1}]]$, e.g., (U, Y_U) is a left or right (weak) V -module. For $v \in V$, we define (cf. (2.8))

$$Y_U^o(v, x) = Y_U(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \in (\text{End } U)[[x, x^{-1}]]. \quad (2.19)$$

Then FHL's proof in [FHL] for Proposition 2.3 again in fact proves the following result, which gives the equivalence between the notion of left V -module and the notion of right V -module:

Proposition 2.7 *Let V be a vertex operator algebra and U a vector space together with a linear map Y_U from V to $(\text{End } U)[[x, x^{-1}]]$. Then (U, Y_U) is a left (weak) V -module if and only if (U, Y_U^o) is a right (weak) V -module.*

Notice that

$$\text{Res}_x x Y^o(\omega, x) = \text{Res}_x x^{-3} Y(\omega, x^{-1}) = \text{Res}_x x Y(\omega, x). \quad (2.20)$$

Then $\omega_1 = L(0)$ is represented by the same operator on U for (U, Y_U) and (U, Y_U^o) . Therefore, the assertion in Proposition 2.7 also includes the grading information.

Let $f(x) = \sum_{n \leq N} a_n x^n \in U((x^{-1}))$ for a vector space U . Then, for any complex number z_0 ,

$$f(x + z_0) := \sum_{n \leq N} a_n (x + z_0)^n = \sum_{n \leq N} \sum_{j \geq 0} \binom{n}{j} a_n z_0^j x^{n-j} \quad (2.21)$$

exists in $U((x^{-1}))$. Let (W, Y_W) be a right weak V -module and z_0 be a complex number. For $v \in V$, we define $Y_W^{(z_0)}(v, x) \in (\text{End } W)[[x, x^{-1}]]$ by

$$Y_W^{(z_0)}(v, x)w = Y_W(v, x + z_0)w = (Y_W(v, y)w)|_{y=x+z_0} = e^{z_0 \frac{d}{dx}}(Y_W(v, x)w) \quad (2.22)$$

for $w \in W$.

Proposition 2.8 *For any right weak V -module (W, Y_W) , the pair $(W, Y_W^{(z_0)})$ is also a right weak V -module.*

Proof. Clearly,

$$Y_W^{(z_0)}(\mathbf{1}, x) = 1, \quad Y_W^{(z_0)}(v, x)w \in W((x^{-1})) \quad (2.23)$$

for $v \in V$, $w \in W$. Then it remains to prove the opposite Jacobi identity.

Let $u, v \in V$. Replacing x_1 and x_2 by $x_1 + z_0$ and $x_2 + z_0$ in the opposite Jacobi identity (2.12) with Y^o being replaced by Y_W , respectively, we get

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(v, x_2 + z_0) Y_W(u, x_1 + z_0) \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(u, x_1 + z_0) Y_W(v, x_2 + z_0) \\ = & (x_2 + z_0)^{-1} \delta \left(\frac{x_1 + z_0 - x_0}{x_2 + z_0} \right) Y_W(Y(u, x_0)v, x_2 + z_0), \end{aligned} \quad (2.24)$$

noting that for $n \in \mathbb{Z}$,

$$(x_1 + z_0 - x_2 - z_0)^n = (x_1 - x_2)^n. \quad (2.25)$$

Notice that

$$(x_2 + z_0)^{-1} \delta \left(\frac{x_1 + z_0 - x_0}{x_2 + z_0} \right) = e^{z_0(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \quad (2.26)$$

because

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) = 0 \quad (2.27)$$

(cf. [Li2]). Then we obtain

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(v, x_2 + z_0) Y_W(u, x_1 + z_0) \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(u, x_1 + z_0) Y_W(v, x_2 + z_0) \\ = & x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2 + z_0). \end{aligned} \quad (2.28)$$

(This is also a generalization of the opposite Jacobi identity for Y_W with $z_0 = 0$.) This proves that $Y_W^{(z_0)}$ satisfies the opposite Jacobi identity. \square

Remark 2.9 Let (W, Y_W) be a right weak V -module on which $L(-1)$ is locally nilpotent, e.g., (W, Y_W) is a right V -module (recall Remark 2.6). In view of (2.17), we have

$$Y_W^{(z_0)}(v, x) = Y_W(v, x + z_0) = e^{-z_0 L(-1)} Y_W(v, x) e^{z_0 L(-1)} \quad (2.29)$$

for $v \in V$. Then $(W, Y_W^{(z_0)})$ carries the structure of a right weak V -module, which is the transported structure through the linear isomorphism $e^{-z_0 L(-1)}$ from (W, Y_W) . This gives a different proof of Proposition 2.8 in this special case and it also proves that $e^{-z_0 L(-1)}$ is a V -isomorphism from (W, Y_W) to $(W, Y_W^{(z_0)})$.

Remark 2.10 Let (W, Y_W) be a left (weak) V -module and z_0 be a complex number. We first get a right (weak) V -module (W, Y_W^o) by Proposition 2.7. In view of Proposition 2.8, $(W, (Y_W^o)^{(z_0)})$ is again a right (weak) V -module. Then using Proposition 2.7 again we get a left (weak) V -module $(W, ((Y_W^o)^{(z_0)})^o)$. The explicit expression of the vertex operator map for this left module is given by

$$\begin{aligned}
& ((Y_W^o)^{(z_0)})^o(v, x)w \\
&= (Y_W^o)^{(z_0)}(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \\
&= Y_W^o(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1} + z_0)w \\
&= Y_W\left(e^{(x^{-1}+z_0)L(1)}(-(x^{-1} + z_0)^{-2})^{L(0)}e^{xL(1)}(-x^{-2})^{L(0)}v, \frac{x}{1+z_0x}\right)w \quad (2.30)
\end{aligned}$$

for $v \in V$, $w \in W$. From (5.3.3) of [FHL] we have

$$x_1^{L(0)}L(1)x_1^{-L(0)} = x_1^{-1}L(1), \quad (2.31)$$

which immediately gives

$$x_1^{L(0)}e^{xL(1)}x_1^{-L(0)} = e^{xx_1^{-1}L(1)}. \quad (2.32)$$

Using this formula we obtain

$$((Y_W^o)^{(z_0)})^o(v, x)w = Y_W\left(e^{-z_0(1+z_0x)L(1)}(1+z_0x)^{-2L(0)}v, \frac{x}{1+z_0x}\right)w. \quad (2.33)$$

Furthermore, if $L(1)$ local nilpotently acts on W , by the conjugation formula (5.2.38) of [FHL] we have (cf. (2.29))

$$((Y_W^o)^{(z_0)})^o(v, x) = e^{-z_0L(1)}Y(v, x)e^{z_0L(1)}. \quad (2.34)$$

Then $e^{z_0L(1)}$ is a V -isomorphism from the resulted left module to the original left module. This can be considered as a special case of the general change-variable ([Z], [H2]).

Notice that for a V -module W , W' is considerably smaller than W^* if W is infinite-dimensional — This turns out to be the case almost all of the times. Now we consider the whole space W^* . Clearly, the definition (2.7) still makes sense if one replaces w' (an element of W') by any element of W^* . Let W be a weak V -module for now and let $v \in V$, $\alpha \in W^*$. We define

$$Y^*(v, x)\alpha \in W^*[[x, x^{-1}]] \quad (2.35)$$

by

$$\langle Y^*(v, x)\alpha, w \rangle = \langle \alpha, Y^o(v, x)w \rangle \quad (2.36)$$

for $w \in W$.

Remark 2.11 Notice that the symbol $Y^*(v, x)$ was used *differently* in [HL1] where the notion $Y^*(v, x)$ defined in [HL1] is the notion $Y^o(v, x)$ in this paper. In our notations, Y^* goes with W^* just as Y' goes with W' , and the notation Y^o indicates that Y^o is an analogue of the classical opposite multiplication.

By Proposition 5.3.1 of [FHL] we have

$$\langle \alpha, Y(v, x)w \rangle = \left\langle Y^* \left(e^{xL(1)} \left(-x^{-2} \right)^{L(0)} v, x^{-1} \right) \alpha, w \right\rangle. \quad (2.37)$$

Since $Y^o(\mathbf{1}, x) = 1$ (recall (2.9)), we have

$$Y^*(\mathbf{1}, x) = 1. \quad (2.38)$$

For $v \in V$, we set

$$Y^*(v, x) = \sum_{n \in \mathbb{Z}} v_n^* x^{-n-1}. \quad (2.39)$$

When $v = \omega$ (the Virasoro element), we set

$$Y^*(\omega, x) = \sum_{n \in \mathbb{Z}} L^*(n) x^{-n-2}, \quad (2.40)$$

i.e., $L^*(n) = \omega_{n+1}^*$ for $n \in \mathbb{Z}$. Then (by (2.10))

$$\langle L^*(n) \alpha, w \rangle = \langle \alpha, L(-n)w \rangle \quad (2.41)$$

for $n \in \mathbb{Z}$, $\alpha \in W^*$, $w \in W$.

Remark 2.12 By (2.11) and (2.36) we have

$$\langle Y^*(v, x) \alpha, w \rangle = \langle \alpha, Y^o(v, x)w \rangle \in \mathbb{C}((x^{-1})) \quad (2.42)$$

for $w \in W$, i.e.,

$$Y^*(v, x) \alpha \in \text{Hom}(W, \mathbb{C}((x^{-1}))). \quad (2.43)$$

In terms of components, for any $w \in W$ we have

$$\langle v_n^* \alpha, w \rangle = 0 \quad \text{for } n \text{ sufficiently small.} \quad (2.44)$$

The opposite Jacobi identity (2.12) directly gives the following Jacobi identity in terms of matrix-coefficients:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \langle Y^*(u, x_1) Y^*(v, x_2) \alpha, w \rangle \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \langle Y^*(v, x_2) Y^*(u, x_1) \alpha, w \rangle \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \langle Y^*(Y(u, x_0)v, x_2) \alpha, w \rangle \end{aligned} \quad (2.45)$$

for $u, v \in V$, $\alpha \in W^*$, $w \in W$.

Remark 2.13 From (2.45) we have the following “Jacobi identity”

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y^*(u, x_1) Y^*(v, x_2) \alpha - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y^*(v, x_2) Y^*(u, x_1) \alpha \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y^*(Y(u, x_0) v, x_2) \alpha. \end{aligned} \quad (2.46)$$

It is important to notice that *unlike the usual Jacobi identity, (2.46) is not algebraic* in the sense that the coefficient of each monomial $x_0^r x_1^s x_2^l$ in each of the three terms is in general an infinite sum in terms of u_m^* and v_n^* although it is a well defined element of W^* . The main reason is that $Y^*(v, x) \alpha$ in general involves infinitely many negative powers of x . Because of the failure of the truncation condition, technically speaking (W^*, Y^*) does not carry the structure of a weak V -module. On the other hand, in addition to the Jacobi identity (2.46), (W^*, Y^*) also satisfies the vacuum property $Y^*(\mathbf{1}, x) = 1$ and the $L(-1)$ -derivative property $Y^*(L(-1)v, x) = \frac{d}{dx} Y^*(v, x)$ for $v \in V$, which, as proved in [DLM2] (cf. Remark 2.2), follows from (2.45), the vacuum property and the fact $L(-1)v = v_{-2}\mathbf{1}$.

As an immediate consequence we have (cf. [Li5]):

Corollary 2.14 *For any weak V -module W , W^* is a $g(V)$ -module where $v(n)$ acts as v_n^* for $v \in V, n \in \mathbb{Z}$. In particular, the following commutator formula (cf. (2.6)) holds:*

$$[u_m^*, v_n^*] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}^* \quad (2.47)$$

for $u, v \in V, m, n \in \mathbb{Z}$. \square

From now on we shall freely use the commutator formula on W^* without explicit comments.

The following definition naturally arises from Remark 2.13 (cf. [Li5]):

Definition 2.15 For a weak V -module W , we define $D(W) = (W^*)^{res}$, the largest restricted $g(V)$ -submodule of W^* . That is, $D(W)$ consists of each vector α such that for every $v \in V$,

$$Y^*(v, x) \alpha \in W^*((x)), \quad (2.48)$$

i.e., $v_n^* \alpha = 0$ for n sufficiently large.

As it has been noticed in [Li5], from (2.12) we immediately have:

Proposition 2.16 *The pair $(D(W), Y^*)$ carries the structure of a weak V -module and $D(W)$ is the unique maximal weak V -module in W^* with Y^* being the vertex operator map. \square*

Remark 2.17 Let $v \in V$, $\alpha \in D(W)$. Then there exists $r \in \mathbb{Z}$ such that

$$Y^*(v, x)\alpha \in x^r W^*[[x]].$$

Using (2.11), we get

$$\langle Y^*(v, x)\alpha, w \rangle = \langle \alpha, Y^o(v, x)w \rangle \in x^r \mathbb{C}[[x]] \cap \mathbb{C}((x^{-1})) = x^r \mathbb{C}[x] \subset \mathbb{C}[x, x^{-1}] \quad (2.49)$$

for all $w \in W$. That is, $\langle \alpha, Y^o(v, x)w \rangle$ is a Laurent polynomial in x . In other words, $\langle \alpha, Y^o(v, x)w \rangle$ is a meromorphic function on the sphere CP^1 with only two possible poles at 0 and ∞ . Furthermore, when v is fixed with w being free, the orders of the possible pole at 0 are uniformly bounded. Conversely, if $\alpha \in W^*$ satisfies the above mentioned properties, then $\alpha \in D(W)$.

If $W = \coprod_{h \in \mathbb{C}} W_{(h)}$ is an ordinary V -module, then $W^* = \prod_{h \in \mathbb{C}} W_{(h)}^*$ is a formal completion of W' and Y^* is the natural extension of Y' (cf. [HL1]). Let W be a V -module and let $\alpha \in W^*$. Then from (2.41), α is an eigenvector of $L^*(0)$ with eigenvalue h if and only if $\alpha \in W_{(h)}^* \subset W'$. Therefore, W' is the largest generalized submodule of $D(W)$. Let \mathcal{A} be the class of vertex operator algebras satisfying the condition that $L(0)$ acts semisimply on every weak module, i.e., every weak module is a generalized module. Then we immediately have (cf. [Li5]):

Proposition 2.18 *If V is of class \mathcal{A} and W is a V -module, then $D(W) = W'$. \square*

Suppose that V contains a regular vertex operator subalgebra (with the same Virasoro element) V^0 . Since any weak V -module is a weak V^0 -module, $L(0)$ (the same for both V and V^0) acts semisimply on any weak V -module. Then V is of class \mathcal{A} , so that Proposition 2.18 applies to V .

Remark 2.19 Let (W, Y_W) be a right weak V -module. Define $D(W)$ to be the subspace of W^* , consisting of each α such that $Y^*(v, x)\alpha \in W((x^{-1}))$ for $v \in V$. Then $(D(W), Y^*)$ is the unique maximal right weak V -module in W^* with Y^* being the vertex operator map.

3 The weak $V \otimes V$ -module $\mathcal{D}_{P(z)}(W)$

In this section, given a weak V -module W and a nonzero complex number z , we define a canonical subspace $\mathcal{D}_{P(z)}(W)$ of W^* , consisting of what we call $P(z)$ -linear functionals on W . This space $\mathcal{D}_{P(z)}(W)$ contains $D(W)$ defined in Section 2 as a subspace. We prove that $\mathcal{D}_{P(z)}(W)$ has a natural weak module structure for the tensor product vertex operator algebra $V \otimes V$.

Throughout this section, W is a weak V -module and z is a nonzero complex number. Now we introduce our first key notion.

Definition 3.1 A linear functional α on W is called a $P(z)$ -linear functional if for any $v \in V$, $w \in W$, the formal series $\langle \alpha, Y^o(v, x)w \rangle$, an infinite series in general, absolutely converges to a rational function of x in $\mathbb{C}[x, x^{-1}, (x - z)^{-1}]$ in the domain $|x| > |z|$ such that when v is fixed with w being free, the orders of the possible poles at 0 and z for the associated rational functions are bounded by a fixed integer.

All $P(z)$ -linear functionals on W clearly form a vector space, which we denote by $\mathcal{D}_{P(z)}(W)$.

Notice that a rational function in x with only (two) possible poles at 0 and z can be considered as a meromorphic function with only three possible poles at 0, ∞ and z on the sphere CP^1 . (Recall from Section 2 that $\alpha \in D(W)$ if and only if for any $v \in V$, $w \in W$, the formal series $\langle \alpha, Y^o(v, x)w \rangle$ is a Laurent polynomial, i.e., a meromorphic function on the sphere CP^1 with only two possible poles at 0 and ∞ .)

Remark 3.2 The designation of the notion of $P(z)$ -linear functional is due to a close connection, which will be given in Section 4, between the notion of $\mathcal{D}_{P(z)}(W)$ and Huang and Lepowsky's notion of $P(z)$ -intertwining map defined in [H3] and [HL0-3]. We copy the following information about $P(z)$ from [HL1]: let K be the moduli space of spheres with punctures and local coordinates vanishing at these punctures. Then $P(z)$ is the element of K containing CP^1 with ordered punctures $\infty, z, 0$ and standard local coordinates $\frac{1}{w}, w - z, w$, vanishing at these punctures, respectively. However, in this paper we shall use only the algebraic aspect of vertex operator algebras and one can simply treat $\mathcal{D}_{P(z)}(W)$ as a vector space depending on z and W .

Remark 3.3 Note that if a series $a(x) \in \mathbb{C}((x^{-1}))$ absolutely converges to a rational function in $\mathbb{C}[x, x^{-1}, (x - z)^{-1}]$ in the domain $|x| > R$ for some real number R , then $a(x)$ must absolutely converge in the domain $|x| > |z|$ to the same rational function. Then in Definition 3.1, it is enough to assume that $\langle \alpha, Y^o(v, x)w \rangle$ absolutely converges to a rational function in $\mathbb{C}[x, x^{-1}, (x - z)^{-1}]$ in the domain $|x| > R$ for some real number R .

In the following we give several equivalent definitions of $P(z)$ -linear functional on W .

Lemma 3.4 *Let $\alpha \in W^*$. Then all the following conditions on α are equivalent:*

(a) $\alpha \in \mathcal{D}_{P(z)}(W)$.

(b) For $v \in V$, there exist nonnegative integers l and k such that for every $w \in W$,

$$x^l(x-z)^k \langle \alpha, Y^o(v, x)w \rangle \quad (3.1)$$

is a (finite) polynomial in x .

(c) For $v \in V$, there exist nonnegative integers l and k such that

$$x^l(x-z)^k Y^*(v, x)\alpha \in W^*[[x]]. \quad (3.2)$$

(d) For $v \in V$, there exists a nonnegative integer k such that

$$(x-z)^k Y^*(v, x)\alpha \in W^*((x)). \quad (3.3)$$

Proof. From the definition, $\alpha \in \mathcal{D}_{P(z)}(W)$ if and only if for $v \in V$, there exist $l, k \in \mathbb{N}$ such that for every $w \in W$, the formal series

$$x^l(x-z)^k \langle \alpha, Y^o(v, x)w \rangle$$

absolutely converges in the domain $|x| > |z|$ to a rational function in x without (finite) poles. Notice that a rational function without (finite) poles is simply a polynomial and any series that absolutely converges to a polynomial in the domain $|x| > |z|$ must be a polynomial itself. Then it follows that (a) and (b) are equivalent.

Conditions (c) and (d) are obviously equivalent. To finish the proof we shall show that (b) and (c) are equivalent. Clearly, (b) implies (c). Since $Y^o(v, x)w \in W((x^{-1}))$, from (c) we get

$$x^l(x-z)^k \langle \alpha, Y^o(v, x)w \rangle \in \mathbb{C}[[x]] \cap \mathbb{C}((x^{-1})) = \mathbb{C}[x]. \quad (3.4)$$

This proves that (c) implies (b). \square

Remark 3.5 Since $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$ for $v \in V$, in view of (2.36), (2.37) and Lemma 3.4 (b), $\alpha \in \mathcal{D}_{P(z)}(W)$ if and only if for $v \in V$, there exist $r, s \in \mathbb{N}$ such that for every $w \in W$,

$$x^r(x-z)^s \langle \alpha, Y(v, x^{-1})w \rangle \in \mathbb{C}[x]. \quad (3.5)$$

The following result gives the closeness of $\mathcal{D}_{P(z)}(W)$ under the actions of the component operators v_n^* of each vertex operator $Y^*(v, x)$.

Proposition 3.6 *For $u, v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$, there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{Z}$,*

$$(x-z)^k Y^*(u, x)v_n^* \alpha \in W^*((x)). \quad (3.6)$$

Furthermore, for $v \in V$, $n \in \mathbb{Z}$, $\alpha \in \mathcal{D}_{P(z)}(W)$,

$$v_n^* \alpha \in \mathcal{D}_{P(z)}(W). \quad (3.7)$$

That is, the space $\mathcal{D}_{P(z)}(W)$ is a $g(V)$ -submodule of W^* .

Proof. Let $u, v \in V$, $n \in \mathbb{Z}$. From the commutator formula we have

$$Y^*(u, x)v_n^*\alpha = v_n^*Y^*(u, x)\alpha - \sum_{i \in \mathbb{N}} \binom{n}{i} x^{n-i} Y^*(v_i u, x)\alpha. \quad (3.8)$$

Since $v_i u = 0$ for all but finitely many $i \in \mathbb{N}$, there exists $k \in \mathbb{N}$ (independent of n) such that

$$(x - z)^k Y^*(u, x)\alpha, \quad (x - z)^k Y^*(v_i u, x)\alpha \in W^*((x))$$

for all $i \in \mathbb{N}$. Then using (3.8) we have

$$(x - z)^k Y^*(u, x)v_n^*\alpha \in W^*((x)).$$

It follows immediately from Lemma 3.4 that $v_n^*\alpha \in \mathcal{D}_{P(z)}(W)$. \square

Let $\mathbb{C}(x)$ be the algebra of rational functions in x . Define linear maps $\iota_{x;\infty}$ and $\iota_{x;0}$ (cf. [FLM], [FHL]) from $\mathbb{C}(x)$ to $\mathbb{C}[x, x^{-1}]$ and $\mathbb{C}((x))$, respectively, such that for $f(x) \in \mathbb{C}(x)$, $\iota_{x;\infty}f(x)$ and $\iota_{x;0}f(x)$ are the Laurent series expansions at $x = \infty$ and at $x = 0$, respectively. In particular,

$$\iota_{x;\infty}(x - z_0)^r = \sum_{i \geq 0} \binom{r}{i} (-z_0)^i x^{r-i}; \quad (3.9)$$

$$\iota_{x;0}(x - z_0)^r = \sum_{i \geq 0} \binom{r}{i} (-z_0)^{r-i} x^i \quad (3.10)$$

for $r \in \mathbb{Z}$, $z_0 \in \mathbb{C}^\times$. It is clear that $\iota_{x;\infty}$ and $\iota_{x;0}$ are $\mathbb{C}[x, x^{-1}]$ -linear and one-to-one, and they are algebra homomorphisms. From Definition 3.1, for $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$, $w \in W$, the formal series $\langle \alpha, Y^o(v, x)w \rangle$ lies in the range of $\iota_{x;\infty}$. Then $\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle$ is well defined. Furthermore, $\langle \alpha, Y^o(v, x)w \rangle$ absolutely converges to $\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle$ in the domain $|x| > |z|$. Using the defined ι -maps we define a new vertex operator map Y^R .

Definition 3.7 Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. Then we define

$$Y^R(v, x)\alpha \in W^*[[x, x^{-1}]] \quad (3.11)$$

by

$$\langle Y^R(v, x)\alpha, w \rangle = \iota_{x;0} \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle \quad (3.12)$$

$$= \iota_{x;0} \iota_{x;\infty}^{-1} \langle Y^*(v, x)\alpha, w \rangle \quad (3.13)$$

for $w \in W$.

From the definition we have

$$\iota_{x;0}^{-1} \langle Y^R(v, x)\alpha, w \rangle = \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle = \iota_{x;\infty}^{-1} \langle Y^*(v, x)\alpha, w \rangle. \quad (3.14)$$

Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$ and let l, k be as in Lemma 3.4 (b) or (c). Since $x^l \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle$ is analytic at $x = 0$,

$$\langle x^l Y^R(v, x)\alpha, w \rangle = \iota_{x;0}(x^l \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle) \in \mathbb{C}[[x]] \quad (3.15)$$

for every $w \in W$. Hence

$$x^l Y^R(v, x)\alpha \in W^*[[x]]. \quad (3.16)$$

This proves the following result:

Lemma 3.8 *For $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$,*

$$Y^R(v, x)\alpha \in W^*((x)). \quad \square \quad (3.17)$$

Let $k \in \mathbb{N}$ be as in Lemma 3.4 (b). (Note that we may take the same k for (b), (c) and (d).) Noticing that

$$(x - z)^k \langle \alpha, Y^o(v, x)w \rangle \in \mathbb{C}[x, x^{-1}] \quad (3.18)$$

and that $\iota_{x;\infty}$ and $\iota_{x;0}$ are $\mathbb{C}[x, x^{-1}]$ -linear, from (3.12) we get

$$(x - z)^k \langle Y^R(v, x)\alpha, w \rangle = \iota_{x;0} \iota_{x;\infty}^{-1} (x - z)^k \langle \alpha, Y^o(v, x)w \rangle = (x - z)^k \langle \alpha, Y^o(v, x)w \rangle \quad (3.19)$$

for all $w \in W$. Therefore we have proved the following result:

Lemma 3.9 *For $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$,*

$$(x - z)^k Y^R(v, x)\alpha = (x - z)^k Y^*(v, x)\alpha \quad (3.20)$$

whenever $(x - z)^k Y^(v, x)\alpha \in W^*((x))$ for $k \in \mathbb{N}$. Furthermore*

$$(x - z)^k \langle Y^R(v, x)\alpha, w \rangle = (x - z)^k \langle \alpha, Y^o(v, x)w \rangle \quad (3.21)$$

for all $w \in W$. \square

Set

$$Y^R(v, x) = \sum_{n \in \mathbb{Z}} v_n^R x^{-n-1}. \quad (3.22)$$

For $v = \omega$ (the Virasoro element), following the tradition we set

$$Y^R(\omega, x) = \sum_{n \in \mathbb{Z}} L^R(n) x^{-n-2}. \quad (3.23)$$

For now, v_n^R are only linear maps from $\mathcal{D}_{P(z)}(W)$ to W^* . Next, we shall show that v_n^R are in fact linear endomorphisms of $\mathcal{D}_{P(z)}(W)$. We shall need the following result:

Lemma 3.10 *Let U be a vector space and let*

$$f(x) = \sum_{n \in \mathbb{Z}} f_n x^{-n-1}, \quad g(x) = \sum_{n \in \mathbb{Z}} g_n x^{-n-1} \in U[[x, x^{-1}]]. \quad (3.24)$$

Suppose that either $f(x) \in U((x))$ or $f(x) \in U((x^{-1}))$ and that there exist $k \in \mathbb{N}$ and $z \in \mathbb{C}^\times$ such that

$$(x - z)^k f(x) = (x - z)^k g(x). \quad (3.25)$$

Then for $n \in \mathbb{Z}$,

$$f_n \in \text{linear span } \{g_m \mid m \geq n\} \quad (3.26)$$

if $f(x) \in U((x))$ and

$$f_n \in \text{linear span } \{g_m \mid m \leq n + k\} \quad (3.27)$$

if $f(x) \in U((x^{-1}))$.

Proof. Assume $f(x) \in U((x))$. For $n \in \mathbb{Z}$, let s be a positive integer such that $x^{n+s} f(x) \in U[[x]]$. Then $\text{Res}_x x^{n+j} f(x) = 0$ for $j \geq s$ and furthermore,

$$\text{Res}_x x^{n+i} (x - z)^k f(x) = 0 \quad (3.28)$$

for $i \geq s$. Using this and (3.25) we get

$$\begin{aligned} f_n &= \text{Res}_x x^n f(x) \\ &= \text{Res}_x x^n (-z + x)^{-k} (x - z)^k f(x) \\ &= \text{Res}_x \sum_{i \geq 0} \binom{-k}{i} (-z)^{-k-i} x^{n+i} (x - z)^k f(x) \\ &= \text{Res}_x \sum_{i=0}^{s-1} \binom{-k}{i} (-z)^{-k-i} x^{n+i} (x - z)^k f(x) \\ &= \text{Res}_x \sum_{i=0}^{s-1} \binom{-k}{i} (-z)^{-k-i} x^{n+i} (x - z)^k g(x) \\ &\in \text{linear span } \{g_n, g_{n+1}, \dots, g_{n+k+s-1}\}. \end{aligned} \quad (3.29)$$

If $f(x) \in U((x^{-1}))$, let s' be a positive integer such that $x^{n-s'} f(x) \in x^{-2} U[[x^{-1}]]$, so that

$$\text{Res}_x x^{n-k-i} (x - z)^k f(x) = 0 \quad (3.30)$$

for $i \geq s'$. Then use $(x - z)^{-k}$ instead of $(-z + x)^{-k}$ in (3.29) to complete the proof. \square

In view of Lemmas 3.8 and 3.10 we immediately have:

Corollary 3.11 *Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$ and let $k \in \mathbb{N}$ be such that (3.20) holds. Then for $n \in \mathbb{Z}$,*

$$v_n^R \alpha \in \text{linear span } \{v_m^* \alpha \mid m \geq n\}. \quad (3.31)$$

As an immediate consequence of Corollary 3.11, Proposition 3.6 and Lemma 3.8 we have:

Proposition 3.12 *Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. Then*

$$Y^R(v, x)\alpha \in \left(\mathcal{D}_{P(z)}(W)\right)((x)). \quad (3.32)$$

Remark 3.13 Here is a slightly different proof for Proposition 3.12: Let $k \in \mathbb{N}$ be such that $(x - z)^k Y^*(v, x)\alpha \in W^*((x))$. Then using Proposition 3.6, we get

$$(z - x)^k Y^*(v, x)\alpha \in W^*((x)) \cap \mathcal{D}_{P(z)}(W)[[x, x^{-1}]] = \mathcal{D}_{P(z)}(W)((x)), \quad (3.33)$$

so that

$$(z - x)^{-k} \left[(z - x)^k Y^*(v, x)\alpha \right] \text{ exists} \quad (3.34)$$

in $\mathcal{D}_{P(z)}(W)((x))$. Note that in (3.34), *we must use the left and right brackets* because $(z - x)^{-k} Y^*(v, x)\alpha$ in general does not exist in $W^*[[x, x^{-1}]]$. On the other hand, since $Y^R(v, x)\alpha \in W^*((x))$ (Lemma 3.8), we have

$$(z - x)^{-k} \left((z - x)^k Y^R(v, x)\alpha \right) = \left((z - x)^{-k} (z - x)^k \right) Y^R(v, x)\alpha = Y^R(v, x)\alpha. \quad (3.35)$$

In view of this, multiplying both sides of (3.20) by $(z - x)^{-k}$ we obtain

$$Y^R(v, x)\alpha = (z - x)^{-k} \left[(z - x)^k Y^*(v, x)\alpha \right] \in \left(\mathcal{D}_{P(z)}(W)\right)((x)). \quad (3.36)$$

Remark 3.14 Let $\alpha \in D(W)$ (defined in Section 2). Then by definition

$$Y^*(v, x)\alpha \in W^*((x)) \quad \text{for every } v \in V,$$

hence $\alpha \in \mathcal{D}_{P(z)}(W)$ (with $k = 0$ in Lemma 3.4 (d)). This shows that $D(W) \subset \mathcal{D}_{P(z)}(W)$. Furthermore, by Lemma 3.9,

$$Y^R(v, x)\alpha = Y^*(v, x)\alpha \quad \text{for } v \in V, \alpha \in D(W).$$

Therefore, the pair $(\mathcal{D}_{P(z)}(W), Y^R)$ extends the pair $(D(W), Y^*)$.

By Remark 3.14, $D(W)$ is contained in the intersection of all $\mathcal{D}_{P(z)}(W)$ for $z \in \mathbb{C}^\times$. Conversely, let

$$\alpha \in \mathcal{D}_{P(z_1)}(W) \cap \mathcal{D}_{P(z_2)}(W),$$

where z_1, z_2 are two distinct nonzero complex numbers. Then, for $v \in V$, $w \in W$, the formal series $\langle \alpha, Y^o(v, x)w \rangle$ converges in the domain $|x| > \max\{|z_1|, |z_2|\}$ to a rational function $h(x)$ such that

$$h(x) \in \mathbb{C}[x, x^{-1}, (x - z_1)^{-1}] \cap \mathbb{C}[x, x^{-1}, (x - z_2)^{-1}].$$

Consequently, $h(x) \in \mathbb{C}[x, x^{-1}]$. By Remark 2.17, $\alpha \in D(W)$. Thus we have proved the following simple fact:

Proposition 3.15 *Let W be a weak V -module. Then*

$$\bigcap_{z \in \mathbb{C}^\times} \mathcal{D}_{P(z)}(W) = D(W). \quad \square \quad (3.37)$$

Remark 3.16 Here we consider the existence of products of certain series. Let

$$A(x), B(x) \in \text{Hom}(V, V((x))). \quad (3.38)$$

Then just like one of the three main terms in the Jacobi identity,

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) A(x_1) B(x_2) \quad \text{exists} \quad (3.39)$$

in $(\text{End } V)[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]]$. Furthermore, for any $m, n \in \mathbb{Z}$,

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (z + x_1)^m (z + x_2)^n A(x_1) B(x_2) \quad \text{still exists} \quad (3.40)$$

in $(\text{End } V)[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]]$, or equivalently, for each $l \in \mathbb{Z}$,

$$(x_1 - x_2)^l (z + x_1)^m (z + x_2)^n A(x_1) B(x_2) \quad \text{exists} \quad (3.41)$$

in $(\text{End } V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$. Indeed, when the expression in (3.41) is applied to a vector in V its coefficient of a monomial $x_1^r x_2^s$ is a finite sum by first considering the coefficient of x_2^s and then considering the coefficient of x_1^r . Therefore, one may multiply the expression in (3.41) by $(z + x_1)^p (z + x_2)^q$ for any $p, q \in \mathbb{Z}$ and apply the associativity law ([FLM], Chapter 2).

As our first key result we have:

Theorem 3.17 *The pair $(\mathcal{D}_{P(z)}(W), Y^R)$ carries the structure of a weak V -module.*

Proof. The truncation condition: $Y^R(v, x)\alpha \in \mathcal{D}_{P(z)}(W)((x))$ for $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$ holds (Proposition 3.12) and $Y^R(\mathbf{1}, x) = Y^*(\mathbf{1}, x) = 1$. Then it remains to prove the Jacobi identity.

Let $u, v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. By Proposition 3.6, there exists $k_0 \in \mathbb{N}$ such that for *all* $m \in \mathbb{Z}$,

$$(x_1 - z)^{k_0} Y^*(u, x_1) v_m^* \alpha \in W^*((x_1)), \quad (3.42)$$

hence (recall Lemma 3.9)

$$(x_1 - z)^{k_0} Y^R(u, x_1) Y^*(v, x_2) \alpha = (x_1 - z)^{k_0} Y^*(u, x_1) Y^*(v, x_2) \alpha. \quad (3.43)$$

Let $k' \geq k_0$ be such that

$$(x_2 - z)^{k'} Y^R(v, x_2) \alpha = (x_2 - z)^{k'} Y^*(v, x_2) \alpha \in W^*((x_2)).$$

Then by (3.43),

$$\begin{aligned} & (x_2 - z)^{k'} (x_1 - z)^{k'} Y^R(u, x_1) Y^R(v, x_2) \alpha \\ &= (x_2 - z)^{k'} (x_1 - z)^{k'} Y^*(u, x_1) Y^*(v, x_2) \alpha. \end{aligned} \quad (3.44)$$

Similarly, there exists $k'' \in \mathbb{N}$ such that

$$\begin{aligned} & (x_1 - z)^{k''} (x_2 - z)^{k''} Y^R(v, x_2) Y^R(u, x_1) \alpha \\ &= (x_1 - z)^{k''} (x_2 - z)^{k''} Y^*(v, x_2) Y^*(u, x_1) \alpha. \end{aligned} \quad (3.45)$$

Set $k = k' + k''$. For any $w \in W$, using (3.44), (3.45) and (2.45) we obtain

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (x_1 - z)^k (x_2 - z)^k \langle Y^R(u, x_1) Y^R(v, x_2) \alpha, w \rangle \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (x_1 - z)^k (x_2 - z)^k \langle Y^R(v, x_2) Y^R(u, x_1) \alpha, w \rangle \\ &= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (x_1 - z)^k (x_2 - z)^k \langle Y^*(u, x_1) Y^*(v, x_2) \alpha, w \rangle \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (x_1 - z)^k (x_2 - z)^k \langle Y^*(v, x_2) Y^*(u, x_1) \alpha, w \rangle \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_1 - z)^k (x_2 - z)^k \langle Y^*(Y(u, x_0)v, x_2) \alpha, w \rangle. \end{aligned} \quad (3.46)$$

Let m be *any fixed* integer. Since $u_n v \neq 0$ for only finitely many $n \geq m$, there is a nonnegative integer k_1 (*depending on* m) such that for *all* $n \geq m$,

$$(x_2 - z)^{k_1} Y^*(u_n v, x_2) \alpha \in \mathcal{D}_{P(z)}(W)((x_2)), \quad (3.47)$$

hence (recall Lemma 3.9)

$$(x_2 - z)^{k_1} Y^R(u_n v, x_2) \alpha = (x_2 - z)^{k_1} Y^*(u_n v, x_2) \alpha. \quad (3.48)$$

Then

$$\begin{aligned} & \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_2 - z)^{k_1} \langle Y^R(Y(u, x_0)v, x_2)\alpha, w \rangle \\ &= \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_2 - z)^{k_1} \langle Y^*(Y(u, x_0)v, x_2)\alpha, w \rangle. \end{aligned} \quad (3.49)$$

Choosing $k_1 \geq k$ and then combining (3.46) with (3.49) we obtain

$$\begin{aligned} & \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (x_1 - z)^{k_1} (x_2 - z)^{k_1} \langle Y^R(u, x_1)Y^R(v, x_2)\alpha, w \rangle \\ & - \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (x_1 - z)^{k_1} (x_2 - z)^{k_1} \langle Y^R(v, x_2)Y^R(u, x_1)\alpha, w \rangle \\ &= \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_1 - z)^{k_1} (x_2 - z)^{k_1} \langle Y^R(Y(u, x_0)v, x_2)\alpha, w \rangle. \end{aligned} \quad (3.50)$$

In view of Remark 3.16, we may multiply both sides of (3.50) by $(-z + x_1)^{-k_1}(-z + x_2)^{-k_1}$ to get

$$\begin{aligned} & \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \langle Y^R(u, x_1)Y^R(v, x_2)\alpha, w \rangle \\ & - \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \langle Y^R(v, x_2)Y^R(u, x_1)\alpha, w \rangle \\ &= \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \langle Y^R(Y(u, x_0)v, x_2)\alpha, w \rangle. \end{aligned} \quad (3.51)$$

Since there is no term depending on m in (3.51) except for x_0^m and m is an arbitrarily fixed integer, dropping $\text{Res}_{x_0} x_0^m$ from (3.51) we obtain the desired Jacobi identity for Y^R . This concludes the proof. \square

Since Y^R depends on z , it is necessary, especially in certain situations, for us to use the notation $Y_{P(z)}^R$ for Y^R , and we shall do so from now on.

Remark 3.18 Notice that in the construction of the left weak V -module $(\mathcal{D}_{P(z)}(W), Y_{P(z)}^R)$ we essentially used the right weak V -module (W, Y^o) (recall Definition 3.7). In view of this, let us denote the left weak V -module $(\mathcal{D}_{P(z)}(W), Y_{P(z)}^R)$ by $(\mathcal{D}_{P(z)}(W, Y^o), Y_{P(z)}^R)$. Now, let (U, Y_U) be a right weak V -module. Then we can go through the whole procedure replacing Y^o by Y_U to obtain a left weak V -module denoted by $(\mathcal{D}_{P(z)}(U, Y_U), Y_{P(z)}^R)$. Or, we can first consider the left weak V -module (U, Y_U^o) (Proposition 2.7) and then apply Theorem 3.17 to get the left weak V -module $(\mathcal{D}_{P(z)}(U, Y_U), Y_{P(z)}^R)$, noting that $(Y_U^o)^o = Y_U$ ([FHL], [HL1]). In the definitions of $\mathcal{D}_{P(z)}(U, Y_U)$ and $Y_{P(z)}^R$ and in the corresponding results, $\langle \alpha, Y_U(v, x)w \rangle$ plays the role that has been played by $\langle \alpha, Y^o(v, x)w \rangle$ so far.

Recall that given a left weak V -module (W, Y_W) and a nonzero complex number z , we have a right weak V -module $(W, (Y_W^o)^{(z)})$ (Propositions 2.7 and 2.8). Using this and the linear maps $\iota_{x;\infty}$ and $\iota_{x;0}$ defined earlier we now define another vertex operator map $Y_{P(z)}^L$; a linear map from V to $(\text{End } \mathcal{D}_{P(z)}(W))[[x, x^{-1}]]$.

Definition 3.19 Let W be a left weak V -module and z a nonzero complex number. For $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$, we define

$$Y_{P(z)}^L(v, x)\alpha \in W^*[[x, x^{-1}]] \quad (3.52)$$

by

$$\langle Y_{P(z)}^L(v, x)\alpha, w \rangle = \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, (Y^o)^{(z)}(v, x)w \rangle \quad (3.53)$$

$$= \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x+z)w \rangle \quad (3.54)$$

for $w \in W$.

Since $\langle \alpha, Y^o(v, y)w \rangle$ absolutely converges in the domain $|y| > |z|$ to a rational function in $\mathbb{C}[y, y^{-1}, (y-z)^{-1}]$, the formal series $\langle \alpha, Y^o(v, x+z)w \rangle$ absolutely converges in the domain $|x| > |z|$, $|x+z| > |z|$ to a rational function in $\mathbb{C}[x, x^{-1}, (x+z)^{-1}]$, where the possible poles at $y=0, z$ are transformed to $x=-z, 0$. In particular, $\langle \alpha, Y^o(v, x+z)w \rangle$ absolutely converges in the domain $|x| > 2|z|$ (to the same rational function). Thus, in view of Remark 3.3 and Definitions 3.19 and 3.7 we immediately have:

Lemma 3.20 *Let W be a left weak V -module and z a nonzero complex number. Then*

$$(\mathcal{D}_{P(z)}(W), Y_{P(z)}^L) = (\mathcal{D}_{P(-z)}(W, (Y_W^o)^{(z)}), Y_{P(-z)}^R). \quad \square \quad (3.55)$$

In view of Remark 3.18, combining Lemma 3.20 with Theorem 3.17 we immediately have:

Proposition 3.21 *Let W be a left weak V -module and z a nonzero complex number. Then the pair $(\mathcal{D}_{P(z)}(W), Y_{P(z)}^L)$ carries the structure of a weak V -module. \square*

Following the tradition, for the Virasoro element ω we set

$$Y_{P(z)}^L(\omega, x) = \sum_{n \in \mathbb{Z}} L_{P(z)}^L(n) x^{-n-2}. \quad (3.56)$$

Recall the following delta-function properties ([FLM], [FHL], [HL1]):

$$x_0^{-1} \delta\left(\frac{x-z}{x_0}\right) = x^{-1} \delta\left(\frac{x_0+z}{x}\right), \quad x^{-1} \delta\left(\frac{z+x_0}{x}\right) = z^{-1} \delta\left(\frac{x-x_0}{z}\right) \quad (3.57)$$

and

$$x_0^{-1} \delta\left(\frac{z-x}{x_0}\right) f(x_0, x) = x_0^{-1} \delta\left(\frac{z-x}{x_0}\right) f(z-x, x) \quad (3.58)$$

for $f(x_0, x) \in \text{Hom}(U_1, U_2[x_0, x_0^{-1}, x, x^{-1}])$, where U_1 and U_2 are any vector spaces. (This can be made more general (cf. [FLM]), but this is enough for our purpose.)

Let $f(x) \in \mathbb{C}[x, x^{-1}, (x-z)^{-1}]$. Then we have (cf. [FHL], [HL1])

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x-z}{x_0} \right) \iota_{x;\infty} f(x) - x_0^{-1} \delta \left(\frac{z-x}{-x_0} \right) \iota_{x;0} f(x) \\ &= z^{-1} \delta \left(\frac{x-x_0}{z} \right) (\iota_{x_0;0} f(x_0+z)). \end{aligned} \quad (3.59)$$

In particular,

$$x_0^{-1} \delta \left(\frac{x-z}{x_0} \right) - x_0^{-1} \delta \left(\frac{z-x}{-x_0} \right) = z^{-1} \delta \left(\frac{x-x_0}{z} \right). \quad (3.60)$$

For $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$, $w \in W$, from the definition we have

$$\iota_{x;0}^{-1} \langle Y_{P(z)}^L(v, x) \alpha, w \rangle = \left(\iota_{y;\infty}^{-1} \langle \alpha, Y^o(v, y) w \rangle \right) |_{y=x+z}. \quad (3.61)$$

Recall that

$$\iota_{x;0}^{-1} \langle Y_{P(z)}^R(v, x) \alpha, w \rangle = \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x) w \rangle = \iota_{x;\infty}^{-1} \langle Y^*(v, x) \alpha, w \rangle. \quad (3.62)$$

In view of this and (3.59), we immediately have:

Proposition 3.22 *Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. Then*

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x-z}{x_0} \right) Y^*(v, x) \alpha - x_0^{-1} \delta \left(\frac{z-x}{-x_0} \right) Y_{P(z)}^R(v, x) \alpha \\ &= z^{-1} \delta \left(\frac{x-x_0}{z} \right) Y_{P(z)}^L(v, x_0) \alpha. \end{aligned} \quad (3.63)$$

The following lemma (cf. Lemma 3.9) immediately follows from Proposition 3.22.

Lemma 3.23 *Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. Then*

$$(z+x_0)^l Y_{P(z)}^L(v, x_0) \alpha = \text{Res}_x x_0^{-1} \delta \left(\frac{x-z}{x_0} \right) x^l Y^*(v, x) \alpha \quad (3.64)$$

$$= (z+x_0)^l Y^*(v, x_0+z) \alpha \quad (3.65)$$

if $x^l Y_{P(z)}^R(v, x) \alpha \in W^*[[x]]$ for $l \in \mathbb{N}$. \square

Proposition 3.24 *For $u, v \in V$,*

$$Y_{P(z)}^L(u, x_1) Y_{P(z)}^R(v, x_2) = Y_{P(z)}^R(v, x_2) Y_{P(z)}^L(u, x_1), \quad (3.66)$$

acting on $\mathcal{D}_{P(z)}(W)$.

Proof. Let $\alpha \in \mathcal{D}_{P(z)}(W)$ and let $k \in \mathbb{N}$ be such that

$$(y - z)^k Y_{P(z)}^R(v, y) \alpha = (y - z)^k Y^*(v, y) \alpha. \quad (3.67)$$

By Proposition 3.6 and Lemma 3.9, we may choose k so large that

$$(y - z)^k Y_{P(z)}^R(v, y) Y^*(u, x) \alpha = (y - z)^k Y^*(v, y) Y^*(u, x) \alpha$$

also holds. Furthermore, since both $Y_{P(z)}^R$ and Y^* satisfy the weak commutativity, which follows from the commutator formula, we may choose k such that the following also hold:

$$(x - y)^k Y^*(u, x) Y^*(v, y) = (x - y)^k Y^*(v, y) Y^*(u, x), \quad (3.68)$$

$$(x - y)^k Y_{P(z)}^R(u, x) Y_{P(z)}^R(v, y) = (x - y)^k Y_{P(z)}^R(v, y) Y_{P(z)}^R(u, x). \quad (3.69)$$

Then using (3.63) and all the above identities we obtain

$$\begin{aligned} & z^{-1} \delta \left(\frac{x - x_0}{z} \right) (y - z)^k (x - y)^k Y_{P(z)}^L(u, x_0) Y_{P(z)}^R(v, y) \alpha \\ = & x_0^{-1} \delta \left(\frac{x - z}{x_0} \right) (y - z)^k (x - y)^k Y^*(u, x) Y_{P(z)}^R(v, y) \alpha \\ & - x_0^{-1} \delta \left(\frac{z - x}{-x_0} \right) (y - z)^k (x - y)^k Y_{P(z)}^R(u, x) Y_{P(z)}^R(v, y) \alpha \\ = & x_0^{-1} \delta \left(\frac{x - z}{x_0} \right) (y - z)^k (x - y)^k Y^*(u, x) Y^*(v, y) \alpha \\ & - x_0^{-1} \delta \left(\frac{z - x}{-x_0} \right) (y - z)^k (x - y)^k Y_{P(z)}^R(u, x) Y_{P(z)}^R(v, y) \alpha \\ = & x_0^{-1} \delta \left(\frac{x - z}{x_0} \right) (y - z)^k (x - y)^k Y^*(v, y) Y^*(u, x) \alpha \\ & - x_0^{-1} \delta \left(\frac{z - x}{-x_0} \right) (y - z)^k (x - y)^k Y_{P(z)}^R(v, y) Y_{P(z)}^R(u, x) \alpha \\ = & x_0^{-1} \delta \left(\frac{x - z}{x_0} \right) (y - z)^k (x - y)^k Y_{P(z)}^R(v, y) Y^*(u, x) \alpha \\ & - x_0^{-1} \delta \left(\frac{z - x}{-x_0} \right) (y - z)^k (x - y)^k Y_{P(z)}^R(v, y) Y_{P(z)}^R(u, x) \alpha \\ = & z^{-1} \delta \left(\frac{x - x_0}{z} \right) (y - z)^k (x - y)^k Y_{P(z)}^R(v, y) Y_{P(z)}^L(u, x_0) \alpha. \end{aligned} \quad (3.70)$$

Taking Res_x of (3.70) and then using (3.58) we get

$$\begin{aligned} & (y - z)^k (z + x_0 - y)^k Y_{P(z)}^L(u, x_0) Y_{P(z)}^R(v, y) \alpha \\ = & (y - z)^k (z + x_0 - y)^k Y_{P(z)}^R(v, y) Y_{P(z)}^L(u, x_0) \alpha. \end{aligned} \quad (3.71)$$

In view of Remark 3.16 we may multiply both sides of (3.71) by $(-z + y)^{-k} (z + x_0 - y)^{-k}$, to get

$$Y_{P(z)}^L(u, x_0) Y_{P(z)}^R(v, y) \alpha = Y_{P(z)}^R(v, y) Y_{P(z)}^L(u, x_0) \alpha.$$

This proves (3.66), \square

Next we shall combine the two weak V -module structures $Y_{P(z)}^L$ and $Y_{P(z)}^R$ on $\mathcal{D}_{P(z)}(W)$ into a weak $V \otimes V$ -module structure. Recall ([FHL], Proposition 3.7.1) that $V \otimes V$ has a natural vertex operator algebra structure, where the vertex operator map Y is defined by

$$Y(u \otimes v, x) = Y(u, x) \otimes Y(v, x) \quad \text{for } u, v \in V. \quad (3.72)$$

It was proved ([FHL], Propositions 4.6.1) that for any V -modules W_1 and W_2 , $W_1 \otimes W_2$ has a natural $V \otimes V$ -module structure.

Let $Y_{P(z)}(\cdot, x)$ be the (unique) linear map from $V \otimes V$ to $(\text{End } \mathcal{D}_{P(z)}(W))[[x, x^{-1}]]$ such that

$$Y_{P(z)}(u \otimes v, x) = Y_{P(z)}^L(u, x)Y_{P(z)}^R(v, x) \quad \text{for } u, v \in V. \quad (3.73)$$

Now we present our second main theorem of the paper.

Theorem 3.25 *The pair $(\mathcal{D}_{P(z)}(W), Y_{P(z)})$ carries the structure of a weak $V \otimes V$ -module.*

Proof. The proof of Proposition 4.6.1 of [FHL] in fact proves the following result: If we have two commuting left weak V -module structures Y_1 and Y_2 on a vector space M in the sense that

$$Y_1(u, x_1)Y_2(v, x_2) = Y_2(v, x_2)Y_1(u, x_1) \quad (3.74)$$

for $u, v \in V$, then $Y_1 \otimes Y_2$ gives rise to a left weak $V \otimes V$ -module structure on M . Then it immediately follows from Theorem 3.17 and Propositions 3.24 and 3.21. \square

Remark 3.26 Let W be a weak V -module. Then from Theorem 3.25, $\coprod_{z \in \mathbb{C}^\times} \mathcal{D}_{P(z)}(W)$ is a weak $V \otimes V$ -module. On the other hand, since each $\mathcal{D}_{P(z)}(W)$ is a subspace of W^* , we consider the sum

$$\mathcal{S}(W) = \sum_{z \in \mathbb{C}^\times} \mathcal{D}_{P(z)}(W). \quad (3.75)$$

Recall (Proposition 3.15) that for distinct nonzero complex numbers z_1 and z_2 , we have

$$\mathcal{D}_{P(z_1)}(W) \cap \mathcal{D}_{P(z_2)}(W) = D(W).$$

Thus $\sum_{z \in \mathbb{C}^\times} \mathcal{D}_{P(z)}(W)$ is not a direct sum. Because of this, we need to consider the existences of an extension of all $Y_{Q(z)}^L$ and an extension of all $Y_{Q(z)}^R$. It turns out that all $Y_{P(z)}^R$ can be put together to give a well defined vertex operator action Y^R of V on $\mathcal{S}(W)$, where

$$\langle Y^R(v, x)\alpha, w \rangle = \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle \quad (3.76)$$

for $v \in V$, $\alpha \in \mathcal{S}(W)$, $w \in W$. Then $(\mathcal{S}(W), Y^R)$ carries the structure of a weak V -module. However, all $Y_{P(z)}^L$ do not give a well defined map from V to $\mathcal{S}(W)[[x, x^{-1}]]$. This is because for $v \in V$, $\alpha \in D(W)$, $w \in W$, by definition

$$\begin{aligned}\langle Y_{P(z_1)}^L(v, x)\alpha, w \rangle &= \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x + z_1)w \rangle, \\ \langle Y_{P(z_2)}^L(v, x)\alpha, w \rangle &= \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x + z_2)w \rangle,\end{aligned}$$

so that $Y_{P(z_1)}^L(v, x)\alpha$ and $Y_{P(z_2)}^L(v, x)\alpha$ are generally different. Therefore, $\mathcal{S}(W)$ is a weak V -module, but not a weak $V \otimes V$ -module.

Remark 3.27 Motivated by $\mathcal{S}(W)$ we define a canonical pair $(\mathcal{D}(W), Y^R)$ as follows:

(a) $\alpha \in \mathcal{D}(W)$ if and only if for every $v \in V$, there exists a polynomial $f_v(x)$ such that

$$f_v(x) \langle \alpha, Y^o(v, x)w \rangle \in \mathbb{C}[x] \quad (3.77)$$

for all $w \in W$.

(b) For $v \in V$, $\alpha \in \mathcal{D}(W)$, $Y^R(v, x)\alpha$ is defined by

$$\langle Y^R(v, x)\alpha, w \rangle = \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle \quad (3.78)$$

for $w \in W$.

By slightly modifying the arguments for $(\mathcal{D}_{P(z)}(W), Y^R)$ carrying the structure of a weak V -module, we see that the pair $(\mathcal{D}(W), Y^R)$ carries the structure of a weak V -module. It is clear that $\mathcal{S}(W)$ is a submodule, but we do not know whether $\mathcal{D}(W)$ coincides with $\mathcal{S}(W)$.

4 Decomposition of $\mathcal{D}_{P(z)}(W)$ into irreducible $V \otimes V$ -modules

In this section, for V -modules W_1, W_2 and W , we shall identify a $P(z)$ -intertwining map of type $\binom{W'}{W_1 W_2}$ in the sense of [H3] and [HL0-3] with a $V \otimes V$ -homomorphism from $W_1 \otimes W_2$ into $\mathcal{D}_{P(z)}(W)$. By using Huang and Lepowsky's one-to-one linear correspondences [HL3] between the space of intertwining operators and the space of $P(z)$ -intertwining maps of the same type we obtain canonical linear isomorphisms from the space of intertwining operators of type $\binom{W'}{W_1 W_2}$ to the space of $V \otimes V$ -homomorphisms from $W_1 \otimes W_2$ to $\mathcal{D}_{P(z)}(W)$. When V is regular, we obtain a decomposition of $\mathcal{D}_{P(z)}(W)$ into irreducible $V \otimes V$ -modules. In the case that $W = V$, we obtain an analogue of Peter-Weyl theorem.

A *generalized V -module* [HL1] is a weak V module on which $L(0)$ acts semisimply. That is, a generalized V -module satisfies all the axioms in the notion of a module except the two grading restrictions on homogeneous spaces. A *lower truncated generalized V -module* [H2] is a generalized V -module that also satisfies the lower truncation condition, one of the two grading restrictions on homogeneous spaces. That is, the only difference

between a module and a generalized module is that the homogeneous subspaces of a generalized module could be infinite-dimensional.

Following [FHL] and [HL1], for a vector space U , we set

$$U\{x\} = \left\{ A(x) = \sum_{h \in \mathbb{C}} a_h x^h \mid a_h \in U \quad \text{for } h \in \mathbb{C} \right\}. \quad (4.1)$$

Throughout this section, W, W_1, W_2 and W_3 are generalized V -modules. Now we recall the definition of an intertwining operator from [FHL] and [HL1]:

Definition 4.1 An intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$ is a linear map \mathcal{Y} from $W_1 \otimes W_2$ to $W_3\{x\}$, or equivalently,

$$\begin{aligned} W_1 &\rightarrow (\text{Hom}(W_2, W_3))\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{C}} w_n x^{-n-1} \quad (\text{where } w_n \in \text{Hom}(W_2, W_3)) \end{aligned} \quad (4.2)$$

such that “all the defining properties of a module action that makes sense hold.” That is, for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, we have the lower truncation condition

$$(w_{(1)})_n w_{(2)} = 0 \quad \text{for } n \text{ whose real part is sufficiently large;} \quad (4.3)$$

the $L(-1)$ -derivative property

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}(w_{(1)}, x); \quad (4.4)$$

and the following Jacobi identity

$$\begin{aligned} &x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(v, x_1) \mathcal{Y}(w_{(1)}, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y(v, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y(v, x_0)w_{(1)}, x_2). \end{aligned} \quad (4.5)$$

All intertwining operators of this type clearly form a vector space, denoted by $\mathcal{V}_{W_1 W_2}^{W_3}$.

The following result can be found in [HL2] (cf. [FHL]):

Proposition 4.2 Let W_1, W_2 and W_3 be (ordinary) V -modules. We have the following (canonical) linear isomorphism relations:

$$\mathcal{V}_{W_1 W_2}^{W_3} \cong \mathcal{V}_{W_2 W_1}^{W_3}, \quad \mathcal{V}_{W_1 W_2}^{W_3} \cong \mathcal{V}_{W_1 W'_2}^{W'_3}. \quad (4.6)$$

Notice that in Sections 2 and 3, only integral powers of z were involved. In this section, intertwining operators will play an important role, so that in general complex powers of z will be involved. Let $\mathcal{Y}(\cdot, x)$ be an intertwining operator of type $\binom{W'}{W_1 W_2}$. Since for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ involves complex powers of x , to consider the evaluation “ $\mathcal{Y}(w_{(1)}, z)w_{(2)}$ ” we need to choose a branch of the log function as in [HL1]. Following [HL1] we choose $\log z$ so that

$$\log z = \log |z| + i \arg z \quad \text{with} \quad 0 \leq \arg z < 2\pi, \quad (4.7)$$

and arbitrary values of the log function will be denoted by

$$l_p(z) = \log z + 2p\pi i \quad (4.8)$$

for $p \in \mathbb{Z}$.

For a V -module $W = \coprod_{h \in \mathbb{C}} W_{(h)}$, following [HL1] we define the formal completion

$$\overline{W} = \prod_{h \in \mathbb{C}} W_{(h)}. \quad (4.9)$$

Each $f \in \overline{W}$ can be viewed as a formal sum $\sum_{h \in \mathbb{C}} f_h$, which is a well defined element of $(W')^*$. In view of this we have

$$\overline{W} = (W')^*, \quad \overline{W'} = W^*. \quad (4.10)$$

Furthermore, as noticed in [HL1] the action of a vertex operator $Y(v, x)$ on W can be naturally extended to \overline{W} . Then $Y^*(v, x)$ acting on $W^* (= \overline{W'})$ is the natural extension of $Y'(v, x)$ on W' . Because of this, we shall also use Y for Y^* (as it was done in [HL1]).

Let U_1 and U_2 be vector spaces and let

$$A(x) = \sum_{h \in \mathbb{C}} a_h x^h \in (\text{Hom}(U_1, U_2))\{x\}$$

be such that for every $u_1 \in U_1$, $a_h u_1 = 0$ for all but finitely many h , so that $A(x)u_1$ is a finite sum. Then we define (cf. [HL1])

$$A(e^{l_p(z)}) = A(x)|_{x=e^{l_p(z)}} := \sum_{h \in \mathbb{C}} e^{hl_p(z)} a_h, \quad (4.11)$$

which is a well defined element of $\text{Hom}(U_1, U_2)$. In the case that $U_2 = \mathbb{C}$ and $U_1 = W$, a V -module, we have $A(y) \in W'\{y\}$ and

$$(f(y)A(y))|_{y=e^{l_p(z)}} = f(z)A(e^{l_p(z)}) \quad (4.12)$$

for $f(y) \in \mathbb{C}[y, y^{-1}]$. If \mathcal{Y} is an intertwining operator of type $\binom{W'}{W_1 W_2}$, then (cf. [HL1])

$$\mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} = \mathcal{Y}(w_{(1)}, x)w_{(2)}|_{x=e^{l_p(z)}} \in W^* \quad (4.13)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$.

We recall the following notion of $P(z)$ -intertwining map from [HL1]:

Definition 4.3 [HL1] Let W_1, W_2 and W be generalized V -modules. A $P(z)$ -intertwining map of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$ is a linear map F from $W_1 \otimes W_2$ to $\overline{W'} (= W^*)$ such that the following identity holds for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y(v, x_1) F(w_{(1)} \otimes w_{(2)}) - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) F(w_{(1)} \otimes Y(v, x_1) w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) F(Y(v, x_0) w_{(1)} \otimes w_{(2)}). \end{aligned} \quad (4.14)$$

Note that $Y(v, x_1)$ in the expression $Y(v, x_1) F(w_{(1)} \otimes w_{(2)})$ is really $Y^*(v, x_1)$, the natural extension of $Y'(v, x_1)$ on W' .

All $P(z)$ -intertwining maps of such type clearly form a vector space, denoted by $\mathcal{M}[P(z)]_{W_1 W_2}^{W'}$.

For $p \in \mathbb{Z}$ and for any intertwining operator \mathcal{Y} of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$, we define a linear map (cf. [HL3])

$$\begin{aligned} F_{\mathcal{Y}, p}^{P(z)} : \quad & W_1 \otimes W_2 \rightarrow \overline{W'} (= W^*) \\ & \sum_{j=1}^n w^{1j} \otimes w^{2j} \mapsto \sum_{j=1}^n \mathcal{Y}(w_{(1j)}, e^{l_p(z)}) w_{(2j)} \end{aligned} \quad (4.15)$$

for $w_{(1j)} \in W_1$, $w_{(2j)} \in W_2$. It is clear that $F_{\mathcal{Y}, p}^{P(z)}$ is an intertwining map of the same type. Furthermore, we have ([HL3], Proposition 12.2):

Proposition 4.4 [HL3] Let W_1, W_2 and W be lower truncated generalized V -modules, let z be a nonzero complex number and let $p \in \mathbb{Z}$. Then the correspondence $\mathcal{Y} \mapsto F_{\mathcal{Y}, p}^{P(z)}$ is a linear isomorphism from the space $\mathcal{V}_{W_1 W_2}^{W'}$ of intertwining operators of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$ onto the space $\mathcal{M}[P(z)]_{W_1 W_2}^{W'}$ of $P(z)$ -intertwining maps of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$.

Now we present our main result in this section.

Theorem 4.5 Let W_1, W_2 and W be generalized V -modules and let F be a linear map from $W_1 \otimes W_2$ to $\overline{W'} (= W^*)$. Then F is a $P(z)$ -intertwining map of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$ if and only if

$$F(W_1 \otimes W_2) \subset \mathcal{D}_{P(z)}(W) \quad (4.16)$$

and F is a $V \otimes V$ -homomorphism from $W_1 \otimes W_2$ into $\mathcal{D}_{P(z)}(W)$ (a subspace of W^*). Equivalently,

$$\mathcal{M}[P(z)]_{W_1 W_2}^{W'} = \text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W)). \quad (4.17)$$

In particular,

$$\dim \mathcal{M}[P(z)]_{W_1 W_2}^{W'} = \dim \text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W)). \quad (4.18)$$

Proof. Suppose that F is a $V \otimes V$ -homomorphism from $W_1 \otimes W_2$ into $\mathcal{D}_{P(z)}(W)$. Let $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. Since $F(w_{(1)} \otimes w_{(2)}) \in \mathcal{D}_{P(z)}(W)$, using Proposition 3.22 and the fact that F is a $V \otimes V$ -homomorphism we get

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y(v, x_1) F(w_{(1)} \otimes w_{(2)}) \\ = & x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) Y_{P(z)}^R(v, x_1) F(w_{(1)} \otimes w_{(2)}) + z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_{P(z)}^L(v, x_0) F(w_{(1)} \otimes w_{(2)}) \\ = & x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) F(w_{(1)} \otimes Y(v, x_1) w_{(2)}) + z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) F(Y(v, x_0) w_{(1)} \otimes w_{(2)}) \end{aligned} \quad (4.19)$$

Then F is a $P(z)$ -intertwining map.

Conversely, suppose that F is a $P(z)$ -intertwining map. For any $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, let $k \in \mathbb{N}$ such that $x_0^k Y(v, x_0) w_{(1)} \in W_1[[x_0]]$. Then by taking $\text{Res}_{x_0} x_0^k$ from (4.14) we obtain

$$(x_1 - z)^k Y(v, x_1) F(w_{(1)} \otimes w_{(2)}) = (x_1 - z)^k F(w_{(1)} \otimes Y(v, x_1) w_{(2)}). \quad (4.20)$$

Since $Y(v, x_1) w_{(2)} \in W_2((x_1))$, the right-hand side of (4.20) lies in $W^*((x_1))$, hence

$$(x_1 - z)^k Y(v, x_1) F(w_{(1)} \otimes w_{(2)}) \in W^*((x_1)). \quad (4.21)$$

Therefore $F(w_{(1)} \otimes w_{(2)}) \in \mathcal{D}_{P(z)}(W)$ by Lemma 3.4. Furthermore, by Lemma 3.9 we have

$$(x_1 - z)^k Y_{P(z)}^R(v, x_1) F(w_{(1)} \otimes w_{(2)}) = (x_1 - z)^k Y(v, x_1) F(w_{(1)} \otimes w_{(2)}). \quad (4.22)$$

Combining this with (4.20) we get

$$(x_1 - z)^k Y_{P(z)}^R(v, x_1) F(w_{(1)} \otimes w_{(2)}) = (x_1 - z)^k F(w_{(1)} \otimes Y(v, x_1) w_{(2)}). \quad (4.23)$$

Because

$$Y_{P(z)}^R(v, x_1) F(w_{(1)} \otimes w_{(2)}), \quad F(w_{(1)} \otimes Y(v, x_1) w_{(2)}) \in \mathcal{D}_{P(z)}(W)((x_1)),$$

we can multiply (4.23) by $(-z + x_1)^{-k}$ to obtain

$$Y_{P(z)}^R(v, x_1) F(w_{(1)} \otimes w_{(2)}) = F(w_{(1)} \otimes Y(v, x_1) w_{(2)}). \quad (4.24)$$

Let $l \in \mathbb{N}$ be such that $x_1^l Y(v, x_1) w_{(2)} \in W_2[[x_1]]$. Then by taking $\text{Res}_{x_1} x_1^l$ from (4.14) we obtain

$$\text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) x_1^l Y(v, x_1) F(w_{(1)} \otimes w_{(2)}) = (x_0 + z)^l F(Y(v, x_0) w_{(1)} \otimes w_{(2)}). \quad (4.25)$$

Similarly, let $l' \in \mathbb{N}$ be such that

$$x_1^{l'} Y_{P(z)}^R(v, x_1) F(w_{(1)} \otimes w_{(2)}) \in (\mathcal{D}_{P(z)}(W))[[x_1]].$$

By taking $\text{Res}_{x_1} x_1^{l'}$ from (3.63) with $\alpha = F(w_{(1)} \otimes w_{(2)})$ we obtain

$$\text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) x_1^{l'} Y(v, x_1) F(w_{(1)} \otimes w_{(2)}) = (x_0 + z)^{l'} Y_{P(z)}^L(v, x_0) F(w_{(1)} \otimes w_{(2)}) \quad (4.26)$$

Combing (4.25) with (4.26) we get

$$(x_0 + z)^{l+l'} Y_{P(z)}^L(v, x_0) F(w_{(1)} \otimes w_{(2)}) = (x_0 + z)^{l+l'} F(Y(v, x_0) w_{(1)} \otimes w_{(2)}). \quad (4.27)$$

Again, because

$$Y_{P(z)}^L(v, x_0) F(w_{(1)} \otimes w_{(2)}), \quad F(Y(v, x_0) w_{(1)} \otimes w_{(2)}) \in (\mathcal{D}_{P(z)}(W))((x_0)),$$

we can multiply (4.27) by $(z + x_0)^{-l-l'}$ to obtain

$$Y_{P(z)}^L(v, x_0) F(w_{(1)} \otimes w_{(2)}) = F(Y(v, x_0) w_{(1)} \otimes w_{(2)}). \quad (4.28)$$

It follows from (4.24) and (4.28) that F is a $V \otimes V$ -homomorphism from $W_1 \otimes W_2$ into $\mathcal{D}_{P(z)}(W)$. \square

From Theorem 4.5, the image of any $P(z)$ -intertwining map of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$ is contained in $\mathcal{D}_{P(z)}(W)$. Let \mathcal{Y} be an intertwining operator of type $\begin{pmatrix} W' \\ W_1 W_2 \end{pmatrix}$. It follows from Proposition 4.4 and Theorem 4.5 that

$$\mathcal{Y}(w_{(1)}, e^{l_p(z)}) w_{(2)} \in \mathcal{D}_{P(z)}(W) \quad (4.29)$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $p \in \mathbb{Z}$. Furthermore, the linear map $F_{\mathcal{Y}, p}^{P(z)}$ is a $V \otimes V$ -homomorphism. In view of this, for $p \in \mathbb{Z}$ we obtain a linear map

$$\begin{aligned} F_p^{P(z)} : \quad \mathcal{V}_{W_1 W_2}^{W'} &\rightarrow \text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W)) \\ \mathcal{Y} &\mapsto F_{\mathcal{Y}, p}^{P(z)}. \end{aligned} \quad (4.30)$$

Combining Proposition 4.4 with Theorem 4.5 we immediately have:

Corollary 4.6 *Let W_1, W_2, W be lower truncated generalized V -modules and let $p \in \mathbb{Z}$. Then $F_p^{P(z)}$ is a linear isomorphism from $\mathcal{V}_{W_1 W_2}^{W'}$ onto $\text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W))$. In particular,*

$$\dim \mathcal{V}_{W_1 W_2}^{W'} = \dim \text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W)). \quad (4.31)$$

Recall that $\mathcal{D}_{P(z)}(W)$ is a weak $V \otimes V$ -module in general. We introduce the following notion in terms of ordinary $V \otimes V$ -submodules of $\mathcal{D}_{P(z)}(W)$:

Definition 4.7 A $P(z)$ -linear functional α on W is called a $P(z)$ -representative functional if α generates an ordinary $V \otimes V$ -submodule of $\mathcal{D}_{P(z)}(W)$, on which $L_{P(z)}^L(0)$ or $L_{P(z)}^R(0)$ semisimply acts. Denote by $R_{P(z)}(W)$ the space of all $P(z)$ -representative functionals on W .

Note that on an ordinary $V \otimes V$ -submodule of $\mathcal{D}_{P(z)}(W)$, $L_{P(z)}^L(0)$ semisimply acts if and only if $L_{P(z)}^R(0)$ semisimply acts. With this definition, any $P(z)$ -intertwining map of type $\binom{W'}{W_1 W_2}$ is a $V \otimes V$ -homomorphism from $W_1 \otimes W_2$ into $R_{P(z)}(W)$. Because of the two grading restrictions in the definition of the notion of a module, $R_{P(z)}(W)$ is a generalized $V \otimes V$ -module in general.

For the rest of this section, let S be a (non-canonical) complete set of representatives of equivalence classes of irreducible V -modules. From [FHL], if W_1 and W_2 are irreducible V -modules, $W_1 \otimes W_2$ is an irreducible $V \otimes V$ -module and furthermore, any irreducible $V \otimes V$ -module is isomorphic to such a module. Then $W_1 \otimes W_2$ for $W_1, W_2 \in S$ form a complete set of representatives of equivalence classes of irreducible $V \otimes V$ -modules. Set

$$H_{V,S}^{P(z)}(W) = \coprod_{W_1, W_2 \in S} \mathcal{M}[P(z)]_{W_1 W_2}^{W'} \otimes (W_1 \otimes W_2), \quad (4.32)$$

which is viewed as a generalized $V \otimes V$ -module with the action on the second parts $(W_1 \otimes W_2)$. With Theorem 4.5 we obtain a $V \otimes V$ -homomorphism Ψ_W from $H_{V,S}^{P(z)}(W)$ into $R_{P(z)}(W)$, defined by

$$\Psi_W(F \otimes (w_{(1)} \otimes w_{(2)})) = F(w_{(1)} \otimes w_{(2)}) \quad (4.33)$$

for $F \in \mathcal{M}[P(z)]_{W_1 W_2}^{W'}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ with $W_1, W_2 \in S$.

As we shall need, we discuss certain properties of modules for a tensor product vertex operator algebra. For vertex operator algebras V_1 and V_2 , we naturally consider V_1 and V_2 as subalgebras of $V_1 \otimes V_2$ by identifying V_1 with $V_1 \otimes \mathbb{C}\mathbf{1}$ and V_2 with $\mathbb{C}\mathbf{1} \otimes V_2$, respectively. In this way, any weak $V_1 \otimes V_2$ -module is automatically a weak V_1 -module and a weak V_2 -module and the actions of V_1 and V_2 commute. (See [FHL], Section 4.7 for more details.) We denote by $L^1(n)$ and $L^2(n)$ for $n \in \mathbb{Z}$ the corresponding Virasoro operators. Then $L^1(n) + L^2(n)$ for $n \in \mathbb{Z}$ are the Virasoro operators for $V_1 \otimes V_2$.

Proposition 4.8 *Let V_1 and V_2 be vertex operator algebras, let W be a weak $V_1 \otimes V_2$ -module, and let W_2 be a finitely generated weak V_2 -module, e.g., W_2 is an irreducible V_2 -module. Then $\text{Hom}_{V_2}(W_2, W)$ is a weak V_1 -module with*

$$(Y(v_{(1)}, x)f)(w_{(2)}) = Y(v_{(1)}, x)f(w_{(2)}) \quad (4.34)$$

for $v_{(1)} \in V_1$, $f \in \text{Hom}_{V_2}(W_2, W)$, $w_{(2)} \in W_2$. Furthermore, if W_2 is a V_1 -module and W is a $V_1 \otimes V_2$ -module on which $L^1(0)$ or $L^2(0)$ semisimply acts, then $\text{Hom}_{V_2}(W_2, W)$ is a V_1 -module.

Proof. Since the actions of V_1 and V_2 on W commute, we easily see that

$$Y(v_{(1)}, x)f \in (\text{Hom}_{V_2}(W_2, W))[[x, x^{-1}]] \quad (4.35)$$

for $v_{(1)} \in V_1$, $f \in \text{Hom}_{V_2}(W_1, W)$. Clearly, the vacuum property holds and the Jacobi identity will also hold provided that the truncation condition holds.

Let P be a finite-dimensional subspace of W_2 , which generates W_2 as a weak V_2 -module. Let $v_{(1)} \in V_1$, $f \in \text{Hom}_{V_2}(W_2, W)$. Since $\dim P < \infty$ and

$$(Y(v_{(1)}, x)f)(p) = Y(v_{(1)}, x)f(p) \in W((x)) \quad \text{for every } p \in P,$$

there exists $r \in \mathbb{N}$ such that

$$x^r(Y(v_{(1)}, x)f)(p) = x^r Y(v_{(1)}, x)f(p) \in W[[x]] \quad \text{for all } p \in P. \quad (4.36)$$

Furthermore, for $v_{(2)} \in V_2$ we have

$$\begin{aligned} x_1^r(Y(v_{(1)}, x_1)f)(Y(v_{(2)}, x_2)p) &= Y(v_{(2)}, x_2)x_1^r Y(v_{(1)}, x_1)f(p) \\ &\in W[[x_1, x_2, x_2^{-1}]] \quad \text{for all } p \in P. \end{aligned} \quad (4.37)$$

Since P generates W_2 as a V_2 -module, by repeatedly using (4.37) we get

$$x^r(Y(v_{(1)}, x)f)(w_{(2)}) \in W[[x]] \quad \text{for all } w_{(2)} \in W_2. \quad (4.38)$$

Thus $x^r Y(v_{(1)}, x)f \in (\text{Hom}_{V_2}(W_2, W))[[x]]$. This proves the truncation condition, so $\text{Hom}_{V_2}(W_2, W)$ is a weak V_1 -module.

Now, assume that W_2 is a V_2 -module and W is a $V_1 \otimes V_2$ -module on which $L^1(0)$ or $L^2(0)$ semisimply acts. Since $L^1(0) + L^2(0)$ semisimply acts on W , both $L^1(0)$ and $L^2(0)$ semisimply act on W . Then

$$W = \coprod_{\alpha \in \mathbb{C}} W[\alpha], \quad (4.39)$$

where

$$W[\alpha] = \{w \in W \mid L^1(0)w = \alpha w\}.$$

Clearly, each $W[\alpha]$ is a weak V_2 -submodule of W . On the other hand, we may take the generating space P of W_2 to be a graded subspace with

$$P = P_{(h_1)} \oplus \cdots \oplus P_{(h_m)}, \quad (4.40)$$

where h_1, \dots, h_m are finitely many distinct complex numbers.

Let $f \in \text{Hom}_{V_2}(W_2, W)$. Since $\dim f(P) < \infty$, there are finitely many (distinct) complex numbers $\alpha_1, \dots, \alpha_n$ such that

$$f(P) \subset W[\alpha_1] \oplus \cdots \oplus W[\alpha_n]. \quad (4.41)$$

Since P generates W_2 under the action of V_2 ,

$$f(W_2) \subset W[\alpha_1] \oplus \cdots \oplus W[\alpha_n]. \quad (4.42)$$

Let f_i be the projection of f onto $W[\alpha_i]$ for $1 \leq i \leq n$. Clearly, each f_i is a V_2 -homomorphism, hence $f_i \in \text{Hom}_{V_2}(W_2, W)$. For $1 \leq i \leq n$, we have

$$L(0)f_i = \alpha_i f_i \quad (4.43)$$

because

$$(L(0)f_i)(w_{(2)}) = L^1(0)f_i(w_{(2)}) = \alpha_i f_i(w_{(2)})$$

for $w_{(2)} \in W_2$. This proves that $L(0)$ semisimply acts on $\text{Hom}_{V_2}(W_2, W)$. If $f \in (\text{Hom}_{V_2}(W_2, W))_{(\alpha)}$ and $p \in P_{(h)}$, then

$$(L^1(0) + L^2(0))f(p) = (L(0)f)(p) + f(L(0)p) = (\alpha + h)f(p).$$

That is, $f(p) \in W_{(\alpha+h)}$. Since P generates W_2 under the action of V_2 , through the restriction map we may identify $(\text{Hom}_{V_2}(W_2, W))_{(\alpha)}$ as a subspace of $\sum_{j=1}^m \text{Hom}(P_{(h_j)}, W_{(\alpha+h_j)})$. Then it follows from the finite-dimensionality of P and the two grading restrictions on W that the two grading restrictions on $\text{Hom}_{V_2}(W_2, W)$ also hold. Therefore, $\text{Hom}_{V_2}(W_2, W)$ is a V_1 -module. \square

We also have the following simple fact:

Lemma 4.9 *Let V be a vertex operator algebra, W_j ($j \in J$) be weak V -modules and W be a finitely generated weak V -module. Then as a vector space,*

$$\text{Hom}_V \left(W, \coprod_{j \in J} W_j \right) = \coprod_{j \in J} \text{Hom}_V(W, W_j). \quad (4.44)$$

Proof. If J is a finite index set, it follows easily from the proof of the corresponding classical result. In general, we shall need the assumption on W . It is clear that

$$\coprod_{j \in J} \text{Hom}_V(W, W_j) \subset \text{Hom}_V \left(W, \coprod_{j \in J} W_j \right). \quad (4.45)$$

For the converse, let P be a finite-dimensional generating space of W . Then for every $f \in \text{Hom}_V(W, \coprod_{j \in J} W_j)$, there are finitely many j_1, \dots, j_n such that

$$f(P) \subset W_{j_1} \oplus \dots \oplus W_{j_n}. \quad (4.46)$$

Since P generates W , we have

$$f(W) \subset W_{j_1} \oplus \dots \oplus W_{j_n}. \quad (4.47)$$

Then

$$f \in \text{Hom}_V(W, W_{j_1}) \oplus \dots \oplus \text{Hom}_V(W, W_{j_n}).$$

This proves the converse of (4.45), hence completes the proof. \square

Because modules for vertex operator algebras have finite-dimensional homogeneous subspaces and we work on \mathbb{C} , Schur's Lemma for irreducible modules holds (cf. [FHL], Remark 4.7.1). Then using the proof of the corresponding classical result (see for example [BD], Chapter II, Proposition 1.14, where Lemma 4.9 is used) we have:

Proposition 4.10 *Let V_1 and V_2 be vertex operator algebras and W be a weak $V_1 \otimes V_2$ -module such that W is a direct sum of irreducible (ordinary) V_2 -modules. Let $S_2 = \{M_i \mid i \in I\}$ be a complete set of representatives of equivalence classes of irreducible V_2 -modules. For each $i \in I$, define a linear map*

$$\begin{aligned} \Psi_i : \quad & \text{Hom}_{V_2}(M_i, W) \otimes M_i \rightarrow W \\ & (f, w) \mapsto f(w). \end{aligned} \quad (4.48)$$

Then the natural linear map

$$\Psi = (\Psi_i) : \coprod_{i \in I} \text{Hom}_{V_2}(M_i, W) \otimes M_i \rightarrow W \quad (4.49)$$

is a $V_1 \otimes V_2$ -isomorphism.

If every $V \otimes V$ -module is completely reducible, then $R_{P(z)}(W)$, being a sum of $V \otimes V$ -modules, is completely reducible. Then it follows from Theorem 4.5 and Proposition 4.10 with $V_1 = \mathbb{C}$ and $V_2 = V \otimes V$ that for any V -module W , Ψ_W , defined previously, is a $V \otimes V$ -isomorphism onto $R_{P(z)}(W)$.

The following is a version of a result of [DMZ] about the rationality of the tensor product of rational vertex operator algebras.

Lemma 4.11 *Let V_1 and V_2 be vertex operator algebras such that every module is completely reducible. Then any $V_1 \otimes V_2$ -module W , on which $L^1(0)$ or $L^2(0)$ semisimply acts, is completely reducible.*

Proof. Since $L^1(0)$ and $L^2(0)$ commute, each vector of W is a sum of common eigenvectors for $L^1(0)$ and $L^2(0)$. Let $w \in W$ be a common eigenvector for $L^1(0)$ and $L^2(0)$ with

$$L^1(0)w = h_1w, \quad L^2(0)w = h_2w. \quad (4.50)$$

Then $w \in W_{(h_1+h_2)}$, the $(h_1 + h_2)$ -eigenspace of $L^1(0) + L^2(0)$. Then under the action of $V \otimes \mathbb{C}$ (or $\mathbb{C} \otimes V$), w generates an ordinary V -module, which is completely reducible from the assumption. Thus W is a completely reducible V -module under the action of $V \otimes \mathbb{C}$ (or $\mathbb{C} \otimes V$). Let $\{W_i \mid i \in I\}$ be a complete set of non-isomorphic irreducible V -submodules of W under the action of $V \otimes \mathbb{C}$. In view of Proposition 4.10 we have

$$W = \oplus_{i \in I} \text{Hom}_V(W_i, W) \otimes W_i.$$

Furthermore, in view of Proposition 4.8 $\text{Hom}_V(W_i, W)$ is a V -module under the action of $\mathbb{C} \otimes V$, which is completely reducible. Proposition 4.7.2 of [FHL] states that the tensor product of irreducible modules for factors is an irreducible module for the tensor product vertex operator algebra. Then it follows that W is a completely reducible $V \otimes V$ -module.

□

Remark 4.12 There are certain notions such as regularity [DLM2] and rationality ([Z], [DLM1], [HL1]), which were defined in terms of the complete reducibility of certain types of weak modules. In [HL1], V was defined to be rational if any V -module is completely reducible, if there are only finitely many non-isomorphic irreducible V -modules and if the fusion rule for any triple of V -modules is finite.

If V is rational in the sense of [HL1], then $H_{V,S}^{P(z)}(W)$ is an (ordinary) $V \otimes V$ -module. Furthermore, it follows from Lemma 4.11 that $R_{P(z)}(W)$ is completely reducible. Then using Proposition 4.10 and Theorem 4.5 we immediately have:

Theorem 4.13 *Let V be a vertex operator algebra such that every V -module is completely reducible and let S be a (non-canonical) complete set of representatives of equivalence classes of irreducible V -modules. Then for any V -module W , Ψ_W is a $V \otimes V$ -isomorphism from $H_{V,S}^{P(z)}(W)$ onto $R_{P(z)}(W)$. In particular, if V is rational in the sense of [HL1], the above assertion holds and $R_{P(z)}(W)$ is an (ordinary) $V \otimes V$ -module. \square*

Let us consider a regular vertex operator algebra V , i.e., any weak V -module is a direct sum of irreducible (ordinary) V -modules. It was proved in [DLM2] that $V \otimes V$ is also regular. (Proposition 4.10 with $V_1 = V_2 = V$ together with Proposition 4.7.2 of [FHL] gives another slightly different proof.) Consequently, $\mathcal{D}_{P(z)}(W) = R_{P(z)}(W)$ is a direct sum of irreducible (ordinary) $V \otimes V$ -modules. It follows from Theorem 4.13 that Ψ_W is a $V \otimes V$ -isomorphism onto $\mathcal{D}_{P(z)}(W)$. Furthermore, it was proved ([Li5], Theorem 3.13) that fusion rules for triples of irreducible modules are finite. Thus, $\mathcal{D}_{P(z)}(W)$ is an (ordinary) $V \otimes V$ -module. To summarize we have:

Theorem 4.14 *Let V be a regular vertex operator algebra and let S be a (non-canonical) complete set of representatives of equivalence classes of irreducible V -modules. Then for any V -module W ,*

$$\mathcal{D}_{P(z)}(W) = R_{P(z)}(W), \quad (4.51)$$

$\mathcal{D}_{P(z)}(W)$ is an (ordinary) $V \otimes V$ -module, and the linear map

$$\Psi_W : \coprod_{W_1, W_2 \in S} \mathcal{M}[P(z)]_{W_1 W_2}^{W'} \otimes (W_1 \otimes W_2) \rightarrow \mathcal{D}_{P(z)}(W) \quad (4.52)$$

is a $V \otimes V$ -isomorphism. \square

Next, we consider the special case that $W = V$. It was known (cf. [Li0], Remark 2.9) that $\mathcal{V}_{V W_1}^{W'_2} \cong \text{Hom}_V(W_1, W'_2)$. Then using Schur's lemma, we find that for irreducible V -modules W_1 and W_2 , $\dim \mathcal{V}_{V W_1}^{W'_2} = 1$ if $W_1 \cong W'_2$, and 0 otherwise. In view of Proposition 4.2 we have $\dim \mathcal{V}_{W_1 W'_1}^{V'} = 1$.

Let (W, Y_W) be a V -module. Then Y_W is an intertwining operator of type $\begin{pmatrix} W \\ V W \end{pmatrix}$. As usual, we use Y for Y_W . From [FHL], we have an intertwining operator of type $\begin{pmatrix} W \\ W V \end{pmatrix}$, defined by

$$Y(w, x)v = e^{xL(-1)}Y(v, -x)w \quad (4.53)$$

for $v \in V$, $w \in W$. For convenience, we refer the canonical intertwining operators Y of types $\binom{W}{VW}$ and $\binom{W}{WV}$ as the *standard* intertwining operators. Furthermore, from [HL2] (see also [FHL]), we have an intertwining operator \mathcal{Y} of type $\binom{V'}{WW'}$, defined by

$$\begin{aligned}\langle \mathcal{Y}(w, x)w', v \rangle &= \langle w', Y(e^{xL(1)}e^{\pi i L(0)}x^{-2L(0)}w, x^{-1})v \rangle \\ &= \langle w', e^{x^{-1}L(-1)}Y(v, -x^{-1})e^{xL(1)}e^{\pi i L(0)}x^{-2L(0)}w \rangle \\ &= \langle e^{x^{-1}L(1)}w', Y(v, -x^{-1})e^{xL(1)}e^{\pi i L(0)}x^{-2L(0)}w \rangle\end{aligned}\quad (4.54)$$

for $w \in W$, $w' \in W'$, $v \in V$. Then for an irreducible V -module W we have

$$\mathcal{V}_{WW'}^{V'} = \mathbb{C}\mathcal{Y}. \quad (4.55)$$

Then in view of Theorem 4.13 we immediately have:

Theorem 4.15 *Let V be a vertex operator algebra such that every V -module is completely reducible and let S be as before. Define a linear map*

$$\Phi_V^{P(z)} : \coprod_{W \in S} W \otimes W' \rightarrow R_{P(z)}(V) \subset \mathcal{D}_{P(z)}(V) \quad (4.56)$$

by

$$\langle \Phi_V^{P(z)}(w \otimes w'), v \rangle = \langle e^{z^{-1}L(1)}w', Y(v, -z^{-1})e^{zL(1)}e^{\pi i L(0)}e^{-2L(0)\log z}w \rangle \quad (4.57)$$

(cf. (4.54)) for $w \in W$, $w' \in W'$, $v \in V$. Then $\Phi_V^{P(z)}$ is a $V \otimes V$ -isomorphism onto $R_{P(z)}(V)$. In particular, the above assertion holds if V is regular, or rational in the sense of [HL1]. \square

Remark 4.16 In conformal field theory, the physical Hilbert space is usually a direct sum of tensor product of irreducible modules for the left moving algebra and the corresponding irreducible modules for the right moving algebra. It is very interesting to notice that the $V \otimes V$ -module $R_{P(z)}(V)$ resembles the physical Hilbert space in conformal field theory.

From Theorem 4.13, for any nonzero complex numbers z_1 and z_2 , $R_{P(z_1)}(W)$ and $R_{P(z_2)}(W)$ are isomorphic generalized $V \otimes V$ -modules if every V -module is completely reducible. In the following we shall give a canonical $V \otimes V$ -isomorphism between $\mathcal{D}_{P(z_1)}(W)$ and $\mathcal{D}_{P(z_2)}(W)$ without the complete reducibility assumption on V -modules.

We recall the following conjugation formula from [FHL] (Lemma 5.2.3):

Lemma 4.17 *Let (M, Y_M) be a generalized V -module. Then*

$$x_0^{L(0)}Y_M(v, x)x_0^{-L(0)} = Y_M(x_0^{L(0)}v, xx_0) \quad (4.58)$$

for $v \in V$, where x and x_0 are independent commuting formal variables. Furthermore, when replacing x_0 by a nonzero complex number z , we have

$$e^{L(0)\log z}Y_M(v, x)e^{-L(0)\log z} = Y_M(z^{L(0)}v, zx). \quad (4.59)$$

Notice that for a nonzero complex number z and for $p \in \mathbb{Z}$, the linear endomorphism $e^{l_p(z)L'(0)}$ of W' is naturally extended to the endomorphism $e^{l_p(z)L^*(0)}$ of $W^*(=\overline{W'})$. From (4.59) we have

$$e^{l_p(z)L^*(0)}Y^*(v, x)e^{-l_p(z)L^*(0)} = Y^*(z_0^{L(0)}v, zx) \quad (4.60)$$

on W^* for $v \in V$. Furthermore, we have (cf. Lemma 4.17):

Proposition 4.18 *Let z, z_1 be nonzero complex numbers and let W be a V -module. Then the linear automorphism $e^{l_p(z)L^*(0)}$ of W^* maps $\mathcal{D}_{P(z_1)}(W)$ onto $\mathcal{D}_{P(zz_1)}(W)$ such that*

$$e^{l_p(z)L^*(0)}Y_{P(z_1)}^R(v, x)\alpha = Y_{P(zz_1)}^R(z^{L(0)}v, zx)e^{l_p(z)L^*(0)}\alpha \quad (4.61)$$

$$e^{l_p(z)L^*(0)}Y_{P(z_1)}^L(v, x)\alpha = Y_{P(zz_1)}^L(z^{L(0)}v, zx)e^{l_p(z)L^*(0)}\alpha \quad (4.62)$$

for $v \in V$, $\alpha \in \mathcal{D}_{P(z_1)}(W)$.

Proof. For $v \in V$, $\alpha \in \mathcal{D}_{P(z_1)}(W)$, let $k \in \mathbb{N}$ be such that

$$(x - z_1)^k Y^*(v, x)\alpha \in W^*((x)), \quad (4.63)$$

hence (by Lemma 3.9),

$$(x - z_1)^k Y_{P(z_1)}^R(v, x)\alpha = (x - z_1)^k Y^*(v, x)\alpha. \quad (4.64)$$

Without losing much generality we may assume that v is homogeneous. Using (4.60) and (4.63) we get

$$\begin{aligned} & (x - zz_1)^k Y^*(v, x)e^{l_p(z)L^*(0)}\alpha \\ &= (x - zz_1)^k e^{l_p(z)L^*(0)}Y^*(z^{-L(0)}v, z^{-1}x)\alpha \\ &= z^{k-\text{wt}v}(z^{-1}x - z_1)^k e^{l_p(z)L^*(0)}Y^*(v, z^{-1}x)\alpha \\ &\in W^*((x)). \end{aligned} \quad (4.65)$$

By Lemma 3.4, $e^{l_p(z)L^*(0)}\alpha \in \mathcal{D}_{P(zz_1)}(W)$. Furthermore, by Lemma 3.9,

$$(x - zz_1)^k Y_{P(zz_1)}^R(v, x)e^{l_p(z)L^*(0)}\alpha = (x - zz_1)^k Y^*(v, x)e^{l_p(z)L^*(0)}\alpha. \quad (4.66)$$

Using (4.64), (4.60) and (4.66) we get

$$\begin{aligned} & z^k (x - z_1)^k e^{l_p(z)L^*(0)}Y_{P(z_1)}^R(v, x)\alpha \\ &= z^k (x - z_1)^k e^{l_p(z)L^*(0)}Y^*(v, x)\alpha \\ &= (zx - zz_1)^k Y^*(z^{L(0)}v, zx)e^{l_p(z)L^*(0)}\alpha \\ &= (zx - zz_1)^k Y_{P(zz_1)}^R(z^{L(0)}v, zx)e^{l_p(z)L^*(0)}\alpha. \end{aligned} \quad (4.67)$$

Then multiplying by $z^{-k}(-z_1 + x)^{-k}$ we obtain (4.61).

Since $\alpha \in \mathcal{D}_{P(z_1)}(W)$ and $e^{l_p(z)L^*(0)}\alpha \in \mathcal{D}_{P(zz_1)}(W)$, by Lemma 3.23 there exists $l \in \mathbb{N}$ such that

$$(x_0 + z_1)^l Y_{P(z_1)}^L(v, x_0)\alpha = \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z_1}{x_0} \right) (x_0 + z_1)^l Y^*(v, x_1)\alpha, \quad (4.68)$$

$$\begin{aligned} & (x_0 + zz_1)^l Y_{P(zz_1)}^L(v, x_0) e^{l_p(z)L^*(0)}\alpha \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - zz_1}{x_0} \right) (x_0 + zz_1)^l Y^*(v, x_1) e^{l_p(z)L^*(0)}\alpha. \end{aligned} \quad (4.69)$$

Using (4.68), (4.60) and (4.69) we get

$$\begin{aligned} & z^l (x_0 + z_1)^l e^{l_p(z)L^*(0)} Y_{P(z_1)}^L(v, x_0)\alpha \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z_1}{x_0} \right) z^l (x_0 + z_1)^l e^{l_p(z)L^*(0)} Y^*(v, x_1)\alpha \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z_1}{x_0} \right) z^l (x_0 + z_1)^l Y^*(z^{L(0)}v, x_1) e^{l_p(z)L^*(0)}\alpha \\ = & \text{Res}_{x_1} z^{-1} x_0^{-1} \delta \left(\frac{z^{-1}x_1 - z_1}{x_0} \right) z^l (x_0 + z_1)^l Y^*(z^{L(0)}v, x_1) e^{l_p(z)L^*(0)}\alpha \\ = & \text{Res}_{x_1} (zx_0)^{-1} \delta \left(\frac{x_1 - zz_1}{zx_0} \right) z^l (x_0 + z_1)^l Y^*(z^{L(0)}v, x_1) e^{l_p(z)L^*(0)}\alpha \\ = & z^l (x_0 + z_1)^l Y_{P(zz_1)}^L(z^{L(0)}v, zx_0) e^{l_p(z)L^*(0)}\alpha. \end{aligned} \quad (4.70)$$

Multiplying by $z^{-l}(z_1 + x_0)^{-l}$ we obtain (4.62). \square

By setting $v = \omega$ (the Virasoro element) in (4.61) and (4.62) we obtain

$$e^{l_p(z)L^*(0)} L_{P(z_1)}^R(0) = L_{P(zz_1)}^R(0) e^{l_p(z)L^*(0)}, \quad (4.71)$$

$$e^{l_p(z)L^*(0)} L_{P(z_1)}^L(0) = L_{P(zz_1)}^L(0) e^{l_p(z)L^*(0)}. \quad (4.72)$$

Consequently, $e^{l_p(z)L^*(0)}$ preserves the weight subspaces. By (4.61) and (4.62) again, $e^{l_p(z)L^*(0)}$ maps an ordinary $V \otimes V$ -submodule of $\mathcal{D}_{P(z_1)}(W)$ to an ordinary $V \otimes V$ -submodule of $\mathcal{D}_{P(zz_1)}(W)$. Therefore we have:

Corollary 4.19 *The linear map $e^{l_p(z)L^*(0)}$ maps $R_{P(z_1)}(W)$ onto $R_{P(zz_1)}(W)$. \square*

To achieve our goal we also need the following fact (cf. Remark 2.10):

Lemma 4.20 *Let (M, Y_M) be a generalized V -module (for now). Define*

$$Y_M^z(v, x) = Y_M(z^{L(0)}v, zx)$$

for $v \in V$. Then $(M, Y_M^z(\cdot, x))$ is also a generalized V -module and $e^{l_p(z)L^(0)}$ is a V -isomorphism from $(M, Y_M(\cdot, x))$ onto $(M, Y_M^z(\cdot, x))$.*

Proof. Using the conjugation formula (4.58) for Y_M , we get

$$e^{L(0)\log(z)}Y_M(v, x)e^{-L(0)\log(z)} = Y_M(z^{L(0)}v, zx) = Y_M^z(v, x).$$

Then $(M, Y_M^z(\cdot, x))$ carries the structure of a generalized V -module, which is the transported structure from $(M, Y_M(\cdot, x))$ through the linear isomorphism $e^{l_p(z)L(0)}$, and furthermore, $e^{l_p(z)L(0)}$ is a V -isomorphism from (M, Y_M) onto (M, Y_M^z) . \square

Now we are ready to prove our last result:

Proposition 4.21 *Let W be a V -module, z, z_1 be nonzero complex numbers and let $p \in \mathbb{Z}$. Then the linear map*

$$\sigma_{(p,z,z_1)} := e^{-l_p(z)L_{P(z_1)}^L(0)} e^{-l_p(z)L_{P(z_1)}^R(0)} e^{l_p(z)L^*(0)} \quad (4.73)$$

is a $V \otimes V$ -isomorphism from $R_{P(z_1)}(W)$ onto $R_{P(z_1)}(W)$.

Proof. By definition,

$$Y_{P(z')}(u \otimes v, x) = Y_{P(z')}^L(u, x)Y_{P(z')}^R(v, x) \quad (4.74)$$

for $u, v \in V$ and for any nonzero complex number z' . Using Corollary 4.19, Proposition 4.18 and Lemma 4.20 we get

$$\begin{aligned} \sigma_{(p,z,z_1)}Y_{P(z_1)}(u \otimes v, x) &= \sigma_{(p,z,z_1)}Y_{P(z_1)}^L(u, x)Y_{P(z_1)}^R(v, x) \\ &= Y_{P(z_1)}^L(u, x)Y_{P(z_1)}^R(v, x)\sigma_{(p,z,z_1)} \\ &= Y_{P(z_1)}(u \otimes v, x)\sigma_{(p,z,z_1)}. \quad \square \end{aligned} \quad (4.75)$$

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